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# ПРЕДИСЛОВИЕ

Этот справочник предназначен для студентов высших учебных заведений, инженеров и научных работников. Он содержит таблицы неопределенных интегралов от элементарных функций. В книге помещены в основном интегралы, выражаемые через элементарные функции; для простейших интегралов, не обладающих этим свойством, но часто встречающихся в приложениях, даны представления через специальные функции. Определения этих специальных функций, а также основные свойства элементарных функций, которые могут быть использованы при вычислении интегралов, приведены в приложениях. Другие неопределенные интегралы от элементарных функций можно найти в более полном справочном руководстве [1].

Постоянная интегрирования в правых частях формул для краткости опущена; например, вместо

$$\int \sin x\,dx = -\cos x + C$$

пишем

$$\int \sin x\,dx = -\cos x.$$

Переменные интегрирования $x$, $t$ и параметры $a$, $b$, $c$, $d$ считаются действительными, $\alpha$, $p$, $q$, $r$ — комплексными, $k$, $l$, $m$, $n = 0, 1, 2, \ldots$; остальные ограничения указываются в квадратных скобках после соответствующих формул. Некоторые формулы при определенных значениях параметров теряют смысл. Если эти значения следуют из структуры формулы, то соответствующие разъяснения опускаются. Выражения для интеграла при этих значениях параметров, как правило, даются в последующих формулах.

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[1] *Прудников А.П., Брычков Ю.А., Маричев О.И.* Интегралы и ряды. Элементарные функции. — М.: Наука, 1981; в этой книге помещены также и определенные интегралы. Неопределенные и определенные интегралы от специальных функций можно найти в книгах: *Прудников А.П., Брычков Ю.А., Маричев О.И.* Интегралы и ряды. Специальные функции. — М.: Наука, 1983; *Прудников А.П., Брычков Ю.А., Маричев О.И.* Интегралы и ряды. Дополнительные главы. — М.: Наука, 1985.

# 1. ВВЕДЕНИЕ

**1.1.** Основные интегралы.

**1.** $\int x^p\,dx = \dfrac{x^{p+1}}{p+1}$ $\qquad [p \neq -1]$.

**2.** $\int \dfrac{dx}{x} = \ln|x|$.

3–7: $[a \neq 0]$

**3.** $\int \dfrac{dx}{x^2+a^2} = \dfrac{1}{a}\,\operatorname{arctg}\dfrac{x}{a}$.

**4.** $\int \dfrac{dx}{x^2-a^2} = \dfrac{1}{2a}\ln\left|\dfrac{x-a}{x+a}\right|$.

**5.** $\int \dfrac{dx}{\sqrt{x^2+a^2}} = \ln\left(x+\sqrt{x^2+a^2}\,\right)$.

**6.** $\int \dfrac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\,\right|$.

**7.** $\int \dfrac{dx}{\sqrt{a^2-x^2}} = \arcsin\dfrac{x}{|a|}$.

**8.** $\int e^x\,dx = e^x$.

**9.** $\int a^x\,dx = \dfrac{a^x}{\ln a}$ $\qquad [a>0,\ a\neq 1]$.

**10.** $\int \operatorname{sh} x\,dx = \operatorname{ch} x$.

**11.** $\int \dfrac{dx}{\operatorname{sh} x} = \ln\left|\operatorname{th}\dfrac{x}{2}\right|$.

**12.** $\int \dfrac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x$.

**13.** $\int \operatorname{ch} x\,dx = \operatorname{sh} x$.

**14.** $\int \dfrac{dx}{\operatorname{ch} x} = 2\ \operatorname{arctg} e^x$.

**15.** $\int \dfrac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x$.

**16.** $\int \operatorname{th} x\,dx = \ln\ \operatorname{ch} x$.

**17.** $\int \operatorname{cth} x\,dx = \ln|\operatorname{sh} x|$.

**18.** $\int \sin x\,dx = -\cos x$.

**19.** $\int \dfrac{dx}{\sin x} = \ln\left|\operatorname{tg}\dfrac{x}{2}\right|$.

**20.** $\int \dfrac{dx}{\sin^2 x} = -\operatorname{ctg} x$.

**21.** $\int \cos x\,dx = \sin x$.

**22.** $\int \dfrac{dx}{\cos x} = \ln\left|\operatorname{tg}\left(\dfrac{\pi}{4}+\dfrac{x}{2}\right)\right|$.

**23.** $\int \dfrac{dx}{\cos^2 x} = \operatorname{tg} x$.

**24.** $\int \dfrac{\sin x}{\cos^2 x}\,dx = \dfrac{1}{\cos x}$.

**25.** $\int \dfrac{\cos x}{\sin^2 x}\,dx = -\dfrac{1}{\sin x}$.

**26.** $\int \operatorname{tg} x\,dx = -\ln|\cos x|$.

**27.** $\int \operatorname{ctg} x\,dx = \ln|\sin x|$.

**1.2.** Общие формулы.

**1.** $\int (af(x)+bg(x))\,dx = a\int f(x)\,dx + b\int g(x)\,dx$.

**2.** $\int \dfrac{df(x)}{dx}\,dx = f(x)$.

**3.** $\int \dfrac{df(x)}{dx}\,g(x)\,dx = f(x)g(x) - \int f(x)\dfrac{dg(x)}{dx}\,dx$

[интегрирование по частям].

**4.** $\int f(x)\,dx = \int f(g(y))g'(y)dy$.

[интегрирование подстановкой; $x = g(y)$]

Дифференцирование неопределенного интеграла:

**5.** $\dfrac{d}{dx}\int f(x)\,dx = f(x)$.

## 2. АЛГЕБРАИЧЕСКИЕ ФУНКЦИИ

### 2.1. Интегралы вида $\int x^p(ax^r+b)^q\,dx$.

1. $$\int x^p(ax^r+b)^q\,dx = \frac{x^{p+1}(ax^r+b)^q}{qr+p+1} + \frac{qrb}{qr+p+1}\int x^p(ax^r+b)^{q-1}\,dx.$$

2. $$= -\frac{x^{p+1}(ax^r+b)^{q+1}}{(q+1)rb} + \frac{qr+r+p+1}{(q+1)rb}\int x^p(ax^r+b)^{q+1}\,dx.$$

3. $$= \frac{x^{p+1}(ax^r+b)^q}{p+1} - \frac{qra}{p+1}\int x^{p+r}(ax^r+b)^{q-1}\,dx.$$

4. $$= \frac{x^{p-r+1}(ax^r+b)^{q+1}}{(q+1)ra} - \frac{p-r+1}{(q+1)ra}\int x^{p-r}(ax^r+b)^{q+1}\,dx.$$

5. $$= \frac{x^{p-r+1}(ax^r+b)^{q+1}}{(qr+p+1)a} - \frac{(p-r+1)b}{(qr+p+1)a}\int x^{p-r}(ax^r+b)^q\,dx.$$

6. $$= \frac{x^{p+1}(ax^r+b)^{q+1}}{(p+1)b} - \frac{(qr+r+p+1)a}{(p+1)b}\int x^{p+r}(ax^r+b)^q\,dx.$$

7. $$\int x^p(ax^r+b)^n\,dx = \sum_{k=0}^{n} C_n^k \frac{a^k b^{n-k} x^{p+kr+1}}{p+kr+1}.$$

8. $$\int \frac{x^p\,dx}{(ax^r+b)^q} = \frac{x^{p+1}}{(q-1)rb(ax^r+b)^{q-1}} + \frac{(q-1)r-p-1}{(q-1)rb}\int \frac{x^p\,dx}{(ax^r+b)^{q-1}}.$$

9. $$= -\frac{x^{p-r+1}}{(q-1)ra(ax^r+b)^{q-1}} + \frac{p-r+1}{(q-1)ra}\int \frac{x^{p-r}\,dx}{(ax^r+b)^{q-1}}.$$

10. $$\int \frac{x^{r-1}\,dx}{(ax^r+b)^q} = -\frac{1}{(q-1)ra(ax^r+b)^{q-1}}.$$

11. $$\int \frac{x^{r-1}\,dx}{ax^r+b} = \frac{1}{ra}\ln|ax^r+b|.$$

12. $$\int \frac{x^r\,dx}{ax^r+b} = \frac{x}{a} - \frac{b}{a}\int \frac{dx}{ax^r+b}.$$

13. $$\int \frac{dx}{(ax^r+b)^q} = \frac{x}{(q-1)rb(ax^r+b)^{q-1}} + \frac{(q-1)r-1}{(q-1)rb}\int \frac{dx}{(ax^r+b)^{q-1}}.$$

14. $$\int \frac{dx}{x^p(ax^r+b)^q} = \frac{1}{(q-1)rbx^{p-1}(ax^r+b)^{q-1}} + \\ + \frac{p+(q-1)r-1}{(q-1)rb}\int \frac{dx}{x^p(ax^r+b)^{q-1}}.$$

**15.** $$\int \frac{dx}{x^p(ax^r+b)} = -\frac{1}{(p-1)bx^{p-1}} - \frac{a}{b}\int \frac{dx}{x^{p-r}(ax^r+b)}.$$

**16.** $$\int \frac{dx}{x(ax^r+b)} = \frac{1}{rb}\ln\left|\frac{x^r}{ax^r+b}\right|.$$

### 2.2. Интегралы вида $\int \frac{x^m\,dx}{x^n+a^n}$.

**1.** $$\int \frac{dx}{x^{2n}+a^{2n}} = -\frac{1}{2na^{2n-1}}\sum_{k=0}^{n-1}\cos\frac{2k+1}{2n}\pi\ln\left(x^2-2ax\cos\frac{2k+1}{2n}\pi+a^2\right) + \frac{1}{na^{2n-1}}\sum_{k=0}^{n-1}\sin\frac{2k+1}{2n}\pi\operatorname{arctg}\frac{x-a\cos\frac{2k+1}{2n}\pi}{a\sin\frac{2k+1}{2n}\pi}.$$

**2.** $$\int \frac{dx}{x^{2n+1}+a^{2n+1}} = \frac{1}{(2n+1)a^{2n}}\ln|x+a| - \frac{1}{(2n+1)a^{2n}}\sum_{k=0}^{n-1}\cos\frac{2k+1}{2n+1}\pi\ln\left(x^2-2ax\cos\frac{2k+1}{2n+1}\pi+a^2\right) + \frac{2}{(2n+1)a^{2n}}\sum_{k=0}^{n-1}\sin\frac{2k+1}{2n+1}\pi\operatorname{arctg}\frac{x-a\cos\frac{2k+1}{2n+1}\pi}{a\sin\frac{2k+1}{2n+1}\pi}.$$

**3.** $$\int \frac{x^m\,dx}{x^{2n}+a^{2n}} = -\frac{1}{2na^{2n-m-1}}\times \sum_{k=0}^{n-1}\cos\frac{(m+1)(2k+1)\pi}{2n}\ln\left(x^2-2ax\cos\frac{2k+1}{2n}\pi+a^2\right) + \frac{1}{na^{2n-m-1}}\sum_{k=0}^{n-1}\sin\frac{(m+1)(2k+1)\pi}{2n}\operatorname{arctg}\frac{x-a\cos\frac{2k+1}{2n}\pi}{a\sin\frac{2k+1}{2n}\pi}$$

$[1 \leqslant m \leqslant 2(n-1)]$.

**4.** $$\int \frac{x^m\,dx}{x^{2n+1}+a^{2n+1}} = \frac{(-1)^m}{(2n+1)a^{2n-m}}\ln|x+a| - \frac{1}{(2n+1)a^{2n-m}}\sum_{k=0}^{n-1}\cos\frac{(m+1)(2k+1)\pi}{2n+1}\ln\left(x^2-2ax\cos\frac{2k+1}{2n+1}\pi+a^2\right) + \frac{2}{(2n+1)a^{2n-m}}\sum_{k=0}^{n-1}\sin\frac{(m+1)(2k+1)\pi}{2n+1}\operatorname{arctg}\frac{x-a\cos\frac{2k+1}{2n+1}\pi}{a\sin\frac{2k+1}{2n+1}\pi}$$

$[1 \leqslant m \leqslant 2n-1]$.

## 2.3. Интегралы вида $\int \frac{x^m\,dx}{x^n - a^n}$.

**1.** $$\int \frac{dx}{x^{2n} - a^{2n}} = \frac{1}{2na^{2n-1}} \ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2na^{2n-1}} \sum_{k=1}^{n-1} \cos\frac{k\pi}{n} \ln\left(x^2 - 2ax\cos\frac{k\pi}{n} + a^2\right) - \frac{1}{na^{2n-1}} \sum_{k=1}^{n-1} \sin\frac{k\pi}{n} \operatorname{arctg} \frac{x - a\cos\frac{k\pi}{n}}{a\sin\frac{k\pi}{n}}.$$

**2.** $$\int \frac{dx}{x^{2n+1} - a^{2n+1}} = \frac{1}{(2n+1)a^{2n}} \ln|x-a| - \frac{1}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \cos\frac{2k+1}{2n+1}\pi \ln\left(x^2 + 2ax\cos\frac{2k+1}{2n+1}\pi + a^2\right) - \frac{2}{(2n+1)a^{2n}} \sum_{k=0}^{n-1} \sin\frac{2k+1}{2n+1}\pi \operatorname{arctg} \frac{x + a\cos\frac{2k+1}{2n+1}\pi}{a\sin\frac{2k+1}{2n+1}\pi}.$$

**3.** $$\int \frac{x^m\,dx}{x^{2n} - a^{2n}} = \frac{1}{2na^{2n-m-1}} \left[\ln|x-a| - (-1)^m \ln|x+a|\right] + \frac{1}{2na^{2n-m-1}} \sum_{k=1}^{n-1} \cos\frac{(m+1)k\pi}{n} \ln\left(x^2 - 2ax\cos\frac{k\pi}{n} + a^2\right) - \frac{1}{na^{2n-m-1}} \sum_{k=1}^{n-1} \sin\frac{(m+1)k\pi}{n} \operatorname{arctg} \frac{x - a\cos\frac{k\pi}{n}}{a\sin\frac{k\pi}{n}} \qquad [1 \leqslant m \leqslant 2(n-1)].$$

**4.** $$\int \frac{x^m\,dx}{x^{2n+1} - a^{2n+1}} = \frac{1}{(2n+1)a^{2n-m}} \ln|x-a| + \frac{(-1)^{m+1}}{(2n+1)a^{2n-m}} \sum_{k=0}^{n-1} \cos\frac{(m+1)(2k+1)\pi}{2n+1} \ln\left(x^2 + 2ax\cos\frac{2k+1}{2n+1}\pi + a^2\right) + \frac{2(-1)^{m+1}}{(2n+1)a^{2n-m}} \sum_{k=0}^{n-1} \sin\frac{(m+1)(2k+1)\pi}{2n+1} \operatorname{arctg} \frac{x + a\cos\frac{2k+1}{2n+1}\pi}{a\sin\frac{2k+1}{2n+1}\pi}$$
$$[1 \leqslant m \leqslant 2n-1].$$

## 2.4. Интегралы вида $\int \frac{x^p\,dx}{(x+a)^q}$.

**1.** $$\int \frac{x^p\,dx}{(x+a)^q} = -\frac{x^p}{(q-1)(x+a)^{q-1}} + \frac{p}{q-1} \int \frac{x^{p-1}\,dx}{(x-a)^{q-1}}.$$

**2.** $\displaystyle\int \frac{x^m\,dx}{(x+a)^q} = -\sum_{k=0}^{m} C_m^k \frac{(-a)^{m-k}}{(q-k-1)(x+a)^{q-k-1}};$

если $k = q-1$, то вместо соответствующего члена в сумме следует взять $C_m^{q-1}(-a)^{m-q+1}\ln|x+a|$.

**3.** $\displaystyle\int \frac{dx}{(x+a)^n} = -\frac{1}{(n-1)(x+a)^{n-1}}.$

**4.** $\displaystyle\int \frac{dx}{x+a} = \ln|x+a|.$

**5.** $\displaystyle\int \frac{x\,dx}{(x+a)^n} = -\frac{1}{(n-2)(x+a)^{n-2}} + \frac{1}{(n-1)(x+a)^{n-1}}.$

**6.** $$\int \frac{x^2\,dx}{(x+a)^n} = -\frac{1}{(n-3)(x+a)^{n-3}} + \frac{2a}{(n-2)(x+a)^{n-2}} - \frac{a^2}{(n-1)(x+a)^{n-1}}.$$

**7.** $$\int \frac{x^3\,dx}{(x+a)^n} = -\frac{1}{(n-4)(x+a)^{n-4}} + \frac{3a}{(n-3)(x+a)^{n-3}} - \frac{3a^2}{(n-2)(x+a)^{n-2}} + \frac{a^3}{(n-1)(x+a)^{n-1}}.$$

**8.** $\displaystyle\int \frac{x^m\,dx}{x+a} = \sum_{k=0}^{m-1} \frac{(-1)^k a^k x^{m-k}}{m-k} + (-a)^m \ln|x+a|.$

**9.** $\displaystyle\int \frac{x\,dx}{x+a} = x - a\ln|x+a|.$

**10.** $\displaystyle\int \frac{x^2\,dx}{x+a} = \frac{x^2}{2} - ax + a^2\ln|x+a|.$

**11.** $\displaystyle\int \frac{x^3\,dx}{x+a} = \frac{x^3}{3} - \frac{ax^2}{2} + a^2x - a^3\ln|x+a|.$

**12.** $$\int \frac{x^m\,dx}{(x+a)^2} = \sum_{k=1}^{m-1} (-1)^k \frac{ka^{k-1}x^{m-k}}{m-k} + \frac{(-1)^{m-1}a^m}{x+a} + (-1)^{m-1}ma^{m-1}\ln|x+a|.$$

**13.** $\displaystyle\int \frac{x\,dx}{(x+a)^2} = \frac{a}{x+a} + \ln|x+a|.$

**14.** $\displaystyle\int \frac{x^2\,dx}{(x+a)^2} = x - \frac{a^2}{x+a} - 2a\ln|x+a|.$

**15.** $\displaystyle\int \frac{x^3\,dx}{(x+a)^2} = \frac{x^2}{2} - 2ax + \frac{a^3}{x+a} + 3a^2\ln|x+a|.$

**16.** $\displaystyle\int \frac{x\,dx}{(x+a)^3} = -\frac{1}{x+a} + \frac{a}{2(x+a)^2}.$

**17.** $\displaystyle\int \frac{x^2\,dx}{(x+a)^3} = \frac{2a}{x+a} - \frac{a^2}{2(x+a)^2} + \ln|x+a|.$

**18.** $\displaystyle\int \frac{x^3\,dx}{(x+a)^3} = x - \frac{3a^2}{x+a} + \frac{a^3}{2(x+a)^2} - 3a\ln|x+a|.$

**19.** $\displaystyle\int \frac{x\,dx}{(x+a)^4} = -\frac{1}{2(x+a)^2} + \frac{a}{3(x+a)^3}.$

**20.** $\displaystyle\int \frac{x^2\,dx}{(x+a)^4} = -\frac{1}{x+a} + \frac{a}{(x+a)^2} - \frac{a^2}{3(x+a)^3}.$

**21.** $\displaystyle\int \frac{x^3\,dx}{(x+a)^4} = \frac{3a}{x+a} - \frac{3a^2}{2(x+a)^2} + \frac{a^3}{3(x+a)^3} + \ln|x+a|.$

## 2.5. Интегралы вида $\displaystyle\int \frac{dx}{x^p(x+a)^q}$.

**1.** $\displaystyle\int \frac{dx}{x^p(x+a)^q} = -\frac{1}{(p-1)ax^{p-1}(x+a)^{q-1}} - \frac{p+q-2}{(p-1)a}\int \frac{dx}{x^{p-1}(x+a)^q}.$

**2.** $\displaystyle\qquad = \frac{1}{(q-1)ax^{p-1}(x+a)^{q-1}} + \frac{p+q-2}{(q-1)a}\int \frac{dx}{x^p(x+a)^{q-1}}.$

**3.** $\displaystyle\int \frac{dx}{x^m(x+a)^n} = \frac{1}{a^{m+n-1}} \sum_{k=0}^{m+n-2} (-1)^{k+1} C_{m+n-2}^k \frac{(x+a)^{m-k-1}}{(m-k-1)x^{m-k-1}};$

если $k = m-1$, то вместо соответствующего члена в сумме следует взять $(-1)^m C_{m+n-2}^{m-1} \ln\left|\dfrac{x+a}{x}\right|$.

**4.** $\displaystyle\int \frac{dx}{x(x+a)^n} = \sum_{k=1}^{n-1} \frac{1}{ka^{n-k}(x+a)^k} - \frac{1}{a^n}\ln\left|\frac{x+a}{x}\right|.$

**5.** $\displaystyle\qquad = \frac{1}{a^n}\sum_{k=1}^{n-1} \frac{(-1)^k}{k} C_{n-1}^k \left(\frac{x}{x+a}\right)^k - \frac{1}{a^n}\ln\left|\frac{x+a}{x}\right|.$

**6.** $\displaystyle\int \frac{dx}{x^2(x+a)^n} = -\frac{1}{ax(x+a)^{n-1}} - \frac{n}{a}\left[\sum_{k=1}^{n-1} \frac{1}{ka^{n-k}(x+a)^k} - \frac{1}{a^n}\ln\left|\frac{x+a}{x}\right|\right].$

**7.** $\displaystyle\int \frac{dx}{x^3(x+a)^n} = \frac{(n+1)x-a}{2a^2x^2(x+a)^{n-1}} + \frac{n(n+1)}{2a^2}\int \frac{dx}{x(x+a)^n}.$

**8.** $\displaystyle\int \frac{dx}{x^m(x+a)} = \sum_{k=1}^{m-1} \frac{(-1)^k}{(m-k)a^k x^{m-k}} + \frac{(-1)^m}{a^m}\ln\left|\frac{x+a}{x}\right|.$

**9.** $\displaystyle\int \frac{dx}{x(x+a)} = \frac{1}{a}\ln\left|\frac{x}{x+a}\right|.$

**10.** $\displaystyle\int \frac{dx}{x^2(x+a)} = -\frac{1}{ax} + \frac{1}{a^2}\ln\left|\frac{x+a}{x}\right|.$

**11.** $\displaystyle\int \frac{dx}{x^3(x+a)} = \frac{1}{a^2x} - \frac{1}{2ax^2} - \frac{1}{a^3}\ln\left|\frac{x+a}{x}\right|.$

**12.** $\displaystyle\int \frac{dx}{x(x+a)^2} = \frac{1}{a}\left(\frac{1}{x+a} - \frac{1}{a}\ln\left|\frac{x+a}{x}\right|\right).$

**13.** $\displaystyle\int \frac{dx}{x^2(x+a)^2} = -\frac{1}{a^2}\left(\frac{1}{x+a} + \frac{1}{x} - \frac{2}{a}\ln\left|\frac{x+a}{x}\right|\right).$

**14.** $\displaystyle\int \frac{dx}{x^3(x+a)^2} = \frac{1}{a^3}\left(\frac{1}{x+a} + \frac{2}{x} - \frac{a}{2x^2} - \frac{3}{a}\ln\left|\frac{x+a}{x}\right|\right).$

**15.** $$\int \frac{dx}{x(x+a)^3} = \frac{1}{a^2(x+a)} + \frac{1}{2a(x+a)^2} - \frac{1}{a^3}\ln\left|\frac{x+a}{x}\right|.$$

**16.** $$\int \frac{dx}{x^2(x+a)^3} = -\frac{1}{2a^2(x+a)^2} - \frac{2}{a^3(x+a)} - \frac{1}{a^3 x} + \frac{3}{a^4}\ln\left|\frac{x+a}{x}\right|.$$

**17.** $$\int \frac{dx}{x^3(x+a)^3} = \frac{1}{a^4}\left[\frac{3}{x+a} + \frac{a}{2(x+a)^2} + \frac{3}{x} - \frac{a}{2x^2} - \frac{6}{a}\ln\left|\frac{x+a}{x}\right|\right].$$

### 2.6. Интегралы вида $\int x^p\left(\frac{x+a}{x+b}\right)^q dx$.

**1.** $$\int x^p\left(\frac{x+a}{x+b}\right)^q dx = \frac{(x+a)(x+b)x^{p-1}}{p+1}\left(\frac{x+a}{x+b}\right)^q - \\ - \frac{q(b-a)+p(b+a)}{p+1}\int x^{p-1}\left(\frac{x+a}{x+b}\right)^q dx - \frac{(p-1)ab}{p+1}\int x^{p-2}\left(\frac{x+a}{x+b}\right)^q dx.$$

**2.** $$\int x\left(\frac{x+a}{x+b}\right)^q dx = \\ = \frac{(x+a)^q}{2(x+b)^{q-2}} + \frac{b(x+a)^q}{(q-1)(x+b)^{q-1}} + \frac{q[(q-1)(a-b)-2b]}{2(q-1)}\int\left(\frac{x+a}{x+b}\right)^{q-1} dx.$$

**3.** $$= \frac{(x+a)^{q+1}}{2(x+b)^{q-1}} - \frac{q(b-a)+a+b}{2}\int\left(\frac{x+a}{x+b}\right)^q dx.$$

**4.** $$\int \frac{(x+a)^p}{(x+b)^q}\,dx = -\frac{(x+a)^p}{(q-1)(x+b)^{q-1}} + \frac{p}{q-1}\int\frac{(x+a)^{p-1}}{(x+b)^{q-1}}\,dx.$$

**5.** $$\int \frac{(x+a)^p}{(x+b)^{p+2}}\,dx = \frac{1}{(p+1)(b-a)}\left(\frac{x+a}{x+b}\right)^{p+1}.$$

**6.** $$\int \frac{(x+a)^p}{(x+b)^{p+3}}\,dx = \frac{(x+a)^{p+1}[x+(p+1)(b-a)+b]}{(p+1)(p+2)(b-a)^2(x+b)^{p+2}}.$$

**7.** $$\int \frac{1}{(x+c)^p}\left(\frac{x+a}{x+b}\right)^q dx = \\ = -\frac{1}{(p-1)(a-c)(b-c)}\left\{\frac{1}{(x+c)^{p-1}}\cdot\frac{(x+a)^{q+1}}{(x+b)^{q-1}} + \right. \\ + [(p-2)(a+b-2c)-q(b-a)]\int\frac{1}{(x+c)^{p-1}}\left(\frac{x+a}{x+b}\right)^q dx + \\ \left. + (p-3)\int\frac{1}{(x+c)^{p-2}}\left(\frac{x+a}{x+b}\right)^q dx\right\}.$$

**8.** $$\int \frac{1}{x+c}\left(\frac{x+a}{x+b}\right)^n dx = \sum_{k=1}^{n-1}\frac{1}{n-k}\left[\left(\frac{a-c}{b-c}\right)^k - 1\right]\left(\frac{x+a}{x+b}\right)^{n-k} - \\ - \left[\left(\frac{a-c}{b-c}\right)^n - 1\right]\ln|x+b| + \left(\frac{a-c}{b-c}\right)^n \ln|x+c|.$$

**9.** $$\int \left(\frac{x+a}{x+b}\right)^n dx = -\sum_{k=0}^{n-2}\frac{n}{(n-k)(n-k-1)}\,\frac{(x+a)^{n-k}}{(x+b)^{n-k-1}} + \\ + n(x+a) - n(b-a)\ln|x+b|.$$

**10.** $$= -\sum_{k=1}^{n-1} C_n^{k+1} \frac{(a-b)^{k+1}}{k(x+b)^k} + x + n(a-b)\ln|x+b|.$$

**11.** $$\int \frac{x+a}{x+b}\,dx = x + (a-b)\ln|x+b|.$$

**12.** $$\int x\frac{x+a}{x+b}\,dx = \frac{x^2}{2} + (a-b)(x - b\ln|x+b|).$$

**13.** $$\int x^2\frac{x+a}{x+b}\,dx = \frac{x^3}{3} + (a-b)\left(\frac{x^2}{2} - bx + b^2\ln|x+b|\right).$$

**14.** $$\int \frac{x+a}{x(x+b)}\,dx = \frac{a}{b}\ln|x| - \frac{a-b}{b}\ln|x+b|.$$

**15.** $$\int \left(\frac{x+a}{x+b}\right)^2 dx = x - \frac{(a-b)^2}{x+b} + 2(a-b)\ln|x+b|.$$

**16.** $$\int x\left(\frac{x+a}{x+b}\right)^2 dx = \frac{x^2}{2} + 2(a-b)x + (a-b)\left[\frac{b(a-b)}{x+b} + (a-3b)\ln|x+b|\right].$$

**17.** $$\int x^2\left(\frac{x+a}{x+b}\right)^2 dx = \frac{x^3}{3} + (a-b)x^2 + \frac{b(a-b)^2x}{x+b} + \\ + (a-b)[(a-3b)(x+b) - 2b(a-2b)\ln|x+b|].$$

**18.** $$\int \frac{1}{x}\left(\frac{x+a}{x+b}\right)^2 dx = \frac{(a-b^2)}{b(x+b)} + \frac{a^2}{b^2}\ln|x| - \left(\frac{a^2}{b^2} - 1\right)\ln|x+b|.$$

**19.** $$\int \frac{(x+a)^2}{x+b}\,dx = \frac{x^2}{2} + (2a-b)x + (a-b)^2\ln|x+b|.$$

**20.** $$\int x\frac{(x+a)^2}{x+b}\,dx = \frac{x^3}{3} + (2a-b)\frac{x^2}{2} + (a-b)^2(x - b\ln|x+b|).$$

**21.** $$\int x^2\frac{(x+a)^2}{x+b}\,dx = \frac{x^4}{4} + (2a-b)\frac{x^3}{3} + (a-b)^2\left(\frac{x^2}{2} - bx + b^2\ln|x+b|\right).$$

**22.** $$\int \frac{(x+a)^2}{x(x+b)}\,dx = x + \frac{1}{b}\,[a^2\ln|x| - (a-b)^2\ln|x+b|].$$

**23.** $$\int \frac{x+a}{(x+b)^2}\,dx = -\frac{a-b}{x+b} + \ln|x+b|.$$

**24.** $$\int x\frac{x+a}{(x+b)^2}\,dx = x + \frac{b(a-b)}{x+b} + (a-2b)\ln|x+b|.$$

**25.** $$\int x^2\frac{x+a}{(x+b)^2}\,dx = \frac{x^2}{2} + (a-2b)x - \frac{b^2(a-b)}{x+b} - b(2a-3b)\ln|x+b|.$$

**26.** $$\int \frac{x+a}{x(x+b)^2}\,dx = \frac{a-b}{b(x+b)} - \frac{a}{b^2}\ln\left|\frac{x+b}{x}\right|.$$

### 2.7. Интегралы вида $\int \frac{x^p\,dx}{(x+a)^q(x+b)^r}$.

**1.** $$\int \frac{x^p\,dx}{(x+a)^q(x+b)^r} = \frac{1}{p-q-r+1}\left\{\frac{x^{p-1}}{(x+a)^{q-1}(x+b)^{r-1}} - \right. \\ \left. - [(p-r)a + (p-q)b]\int \frac{x^{p-1}\,dx}{(x+a)^q(x+b)^r} - (p-1)ab\int \frac{x^{p-2}\,dx}{(x+a)^q(x+b)^r}\right\}.$$

**2.** $\displaystyle \int \frac{x^p\,dx}{(x+a)^m(x+b)^n} =$

$$= -\sum_{k=1}^{m-1} \frac{(-1)^{k+m+p+1}}{k(x+a)^k}\left[\sum_{l=0}^{m-k-1} C_p^l C_{m+n-k-l-2}^{n-1} \frac{a^{p-l}}{(b-a)^{m+n-k-l-1}}\right] -$$
$$- \sum_{k=1}^{n-1} \frac{(-1)^m}{k(x+b)^k}\left[\sum_{l=0}^{n-k-1} C_p^l C_{m+n-k-l-2}^{m-1} \frac{(-b)^{p-l}}{(b-a)^{m+n-k-l-1}}\right] +$$
$$+ (-1)^{m+p+1}\sum_{k=0}^{m-1} C_p^k C_{m+n-k-2}^{n-1} \frac{a^{p-k}}{(b-a)^{m+n-k-1}} \ln|x+a| +$$
$$+ (-1)^m \sum_{k=0}^{n-1} C_p^k C_{m+n-k-2}^{m-1} \frac{(-b)^{p-k}}{(b-a)^{m+n-k-1}} \ln|x+b|.$$

**3.** $$\int \frac{dx}{(x+a)^q(x+b)^r} = -\frac{1}{(r-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} - \frac{q+r-2}{(r-1)(a-b)}\int \frac{dx}{(x+a)^q(x+b)^{r-1}}.$$

**4.** $$= \frac{1}{(q-1)(a-b)(x+a)^{q-1}(x+b)^{r-1}} + \frac{q+r-2}{(q-1)(a-b)}\int \frac{dx}{(x+a)^{q-1}(x+b)^r}.$$

**5.** $$\int \frac{dx}{(x+a)^m(x+b)^n} = \frac{(-1)^{m+n-2}}{(b-a)^{m+n-1}} \sum_{\substack{k=0\\ k\neq n-1}}^{m+n-2} \frac{(-1)^k}{n-k-1} C_{m+n-2}^k \times \left(\frac{x+a}{x+b}\right)^{n-k-1} + \frac{(-1)^{m-1}}{(b-a)^{m+n-1}} C_{m+n-2}^{n-1} \ln\left|\frac{x+a}{x+b}\right|.$$

**6.** $$= \sum_{k=1}^{m-1} \frac{(-1)^{m+k} C_{m+n-k-2}^{n-1}}{k(b-a)^{m+n-k-1}(x+a)^k} - \sum_{k=1}^{n-1} \frac{(-1)^m C_{m+n-k-2}^{m-1}}{k(b-a)^{m+n-k-1}(x+b)^k} + \frac{(-1)^{m+1}}{(b-a)^{m+n-1}} C_{m+n-1}^{n-1} \ln\left|\frac{x+a}{x+b}\right|.$$

**7.** $$\int \frac{dx}{(x+a)(x+b)} = -\frac{1}{a-b}\ln\left|\frac{x+a}{x+b}\right|.$$

**8.** $$\int \frac{x\,dx}{(x+a)(x+b)} = \frac{1}{a-b}\,(a\ln|x+a| - b\ln|x+b|).$$

**9.** $$\int \frac{x^2\,dx}{(x+a)(x+b)} = x + \frac{1}{a-b}\,(b^2\ln|x+b| - a^2\ln|x+a|).$$

**10.** $$\int \frac{dx}{x(x+a)(x+b)} = \frac{1}{ab}\ln|x| + \frac{1}{a(a-b)}\ln|x+a| - \frac{1}{b(a-b)}\ln|x+b|.$$

**11.** $$\int \frac{dx}{x^2(x+a)(x+b)} = -\frac{1}{abx} - \frac{a+b}{a^2b^2}\ln|x| - \frac{1}{a^2(a-b)}\ln|x+a| + \frac{1}{b^2(a-b)}\ln|x+b|.$$

**12.** $$\int \frac{dx}{(x+a)(x+b)^2} = -\frac{1}{(a-b)(x+b)} + \frac{1}{(a-b)^2}\ln\left|\frac{x+a}{x+b}\right|.$$

**13.** $$\int \frac{x\,dx}{(x+a)(x+b)^2} = \frac{b}{(a-b)(x+b)} - \frac{a}{(a-b)^2}\ln\left|\frac{x+a}{x+b}\right|.$$

**14.** $$\int \frac{x^2\,dx}{(x+a)(x+b)^2} = \\ = -\frac{b^2}{(a-b)(x+b)} + \frac{a^2}{(a-b)^2}\ln|x+a| + \frac{b^2-2ab}{(a-b)^2}\ln|x+b|.$$

**15.** $$\int \frac{dx}{x(x+a)(x+b)^2} = \\ = \frac{1}{b(a-b)(x+b)} + \frac{1}{ab^2}\ln|x| - \frac{1}{a(a-b)^2}\ln|x+a| + \frac{2b-a}{b^2(a-b)}\ln|x+b|.$$

**16.** $$\int \frac{dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2}\left(\frac{1}{x+a}+\frac{1}{x+b}\right) + \frac{2}{(a-b)^3}\ln\left|\frac{x+a}{x+b}\right|.$$

**17.** $$\int \frac{x\,dx}{(x+a)^2(x+b)^2} = \frac{1}{(a-b)^2}\left(\frac{a}{x+a}+\frac{b}{x+b}\right) - \frac{a+b}{(a-b)^3}\ln\left|\frac{x+a}{x+b}\right|.$$

**18.** $$\int \frac{x^2\,dx}{(x+a)^2(x+b)^2} = -\frac{1}{(a-b)^2}\left(\frac{a^2}{x+a}+\frac{b^2}{x+b}\right) + \frac{2ab}{(a-b)^3}\ln\left|\frac{x+a}{x+b}\right|.$$

### 2.8. Интегралы вида $\int x^{\pm m}(ax^2+bx+c)^n\,dx$.

**1.** $$\int (ax^2+bx+c)^n\,dx = \frac{1}{2(2n+1)a}\Big[(2ax+b)(ax^2+bx+c)^n - \\ - n(b^2-4ac)\int (ax^2+bx+c)^{n-1}\,dx\Big].$$

**2.** $$= (-1)^n\frac{(n!)^2}{(2n+1)!}\,\frac{2ax+b}{2a}\times \\ \times \sum_{k=0}^{n}(-1)^k C_{2k}^{k}\left(\frac{b^2-4ac}{a}\right)^{n-k}(ax^2+bx+c)^k.$$

**3.** $$\int \frac{(ax^2+bx+c)^n}{x^m}\,dx = -\frac{(ax^2+bx+c)^{n+1}}{(m-1)cx^{m-1}} + \\ + \frac{b(n-m+2)}{c(m-1)}\int \frac{(ax^2+bx+c)^n}{x^{m-1}}\,dx + \frac{a(2n-m+3)}{c(m-1)}\int \frac{(ax^2+bx+c)^n}{x^{m-2}}\,dx.$$

### 2.9. Интегралы вида $\int \frac{dx}{(ax^2+bx+c)^n}$.

**1.** $$\int \frac{dx}{(ax^2+bx+c)^n} = \frac{-2ax-b}{(n-1)(b^2-4ac)(ax^2+bx+c)^{n-1}} - \\ - \frac{2(2n-3)a}{(n-1)(b^2-4ac)}\int \frac{dx}{(ax^2+bx+c)^{n-1}}.$$

**2.** $$= \frac{2ax+b}{2n-1} \times$$
$$\times \sum_{k=0}^{n-2} \frac{2^k(2n-1)(2n-3)\dots(2n-2k-1)a^k}{(n-1)(n-2)\dots(n-k-1)(4ac-b^2)^{k+1}(ax^2+bx+c)^{n-k-1}} +$$
$$+ \frac{(2n-3)!!(2a)^{n-1}}{(n-1)!(4ac-b^2)^{n-1}} \int \frac{dx}{ax^2+bx+c}.$$

**3.** $$\int \frac{dx}{ax^2+bx+c} = \frac{1}{\sqrt{b^2-4ac}} \ln\left|\frac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| \qquad [b^2-4ac>0].$$

**4.** $$= \frac{2}{\sqrt{4ac-b^2}} \operatorname{arctg} \frac{2ax+b}{\sqrt{4ac-b^2}} \qquad [b^2-4ac<0].$$

**5.** $$= -\frac{2}{2ax+b} \qquad [b^2=4ac].$$

**6.** $$= \frac{1}{a(p-q)} \ln\left|\frac{x-p}{x-q}\right|,$$

где $p$, $q$ — действительные корни многочлена $ax^2+bx+c$.

**7.** $$\int \frac{dx}{(ax^2+bx+c)^2} = \frac{2ax+b}{(4ac-b^2)(ax^2+bx+c)} + \frac{2a}{4ac-b^2} \int \frac{dx}{ax^2+bx+c}.$$

**8.** $$\int \frac{dx}{(ax^2+bx+c)^3} = \frac{2ax+b}{2(4ac-b^2)(ax^2+bx+c)^2} +$$
$$+ \frac{3a(2ax+b)}{(4ac-b^2)^2(ax^2+bx+c)} + \frac{6a^2}{(4ac-b^2)^2} \int \frac{dx}{ax^2+bx+c}.$$

## 2.10. Интегралы вида $\int \frac{x^m\,dx}{(ax^2+bx+c)^n}$.

**1.** $$\int \frac{x^m\,dx}{(ax^2+bx+c)^n} = \frac{-x^{m-1}}{(2n-m-1)a(ax^2+bx+c)^{n-1}} +$$
$$+ \frac{(m-1)c}{(2n-m-1)a} \int \frac{x^{m-2}\,dx}{(ax^2+bx+c)^n} - \frac{(n-m)b}{(2n-m-1)a} \int \frac{x^{m-1}\,dx}{(ax^2+bx+c)^n}.$$

**2.** $$\int \frac{x^{2n-1}\,dx}{(ax^2+bx+c)^n} =$$
$$= \frac{1}{a} \int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^{n-1}} - \frac{c}{a} \int \frac{x^{2n-3}\,dx}{(ax^2+bx+c)^n} - \frac{b}{a} \int \frac{x^{2n-2}\,dx}{(ax^2+bx+c)^n}.$$

**3.** $$\int \frac{x\,dx}{(ax^2+bx+c)^n} = -\frac{1}{2(n-1)a(ax^2+bx+c)^{n-1}} - \frac{b}{2a} \int \frac{dx}{(ax^2+bx+c)^n}.$$

**4.** $$= \frac{bx+2c}{(n-1)(b^2-4ac)(ax^2+bx+c)^{n-1}} +$$
$$+ \frac{(2n-3)b}{(n-1)(b^2-4ac)} \int \frac{dx}{(ax^2+bx+c)^{n-1}}.$$

**5.** $$\int \frac{x^2\,dx}{(ax^2+bx+c)^n} = -\frac{(b^2-2ac)(2ax+b)-b(b^2-4ac)}{2(n-1)a^2(b^2-4ac)(ax^2+bx+c)^{n-1}} -$$
$$- \frac{2(2n-3)(b^2-2ac)-2(n-1)(b^2-4ac)}{2(n-1)a(b^2-4ac)} \int \frac{dx}{(ax^2+bx+c)^{n-1}}.$$

**6.** $$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln|ax^2+bx+c| - \\ - \frac{b}{a\sqrt{|b^2-4ac|}} \begin{cases} \dfrac{1}{2}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & [b^2>4ac], \\ \operatorname{arctg}\dfrac{2ax+b}{\sqrt{4ac-b^2}} & [b^2<4ac]. \end{cases}$$

**7.** $$\int \frac{x^2\,dx}{ax^2+bx+c} = \frac{x}{a} - \frac{b}{2a^2}\ln|ax^2+bx+c| + \frac{b^2-2ac}{2a^2}\int\frac{dx}{ax^2+bx+c}.$$

**8.** $$\int \frac{x^3\,dx}{ax^2+bx+c} = \frac{ax^2-2bx}{2a^2} + \\ + \frac{b^2-ac}{2a^3}\ln|ax^2+bx+c| - \frac{b(b^2-3ac)}{2a^3}\int\frac{dx}{ax^2+bx+c}.$$

**9.** $$\int \frac{x\,dx}{(ax^2+bx+c)^2} = \frac{bx+2c}{(b^2-4ac)(ax^2+bx+c)} + \frac{b}{b^2-4ac}\int\frac{dx}{ax^2+bx+c}.$$

**10.** $$\int \frac{x^2\,dx}{(ax^2+bx+c)^2} = \\ = \frac{(2ac-b^2)x-bc}{a(b^2-4ac)(ax^2+bx+c)} - \frac{2c}{b^2-4ac}\int\frac{dx}{ax^2+bx+c}.$$

**11.** $$\int \frac{x^3\,dx}{(ax^2+bx+c)^2} = \frac{c(2ac-b^2)+b(3ac-b^2)x}{a^2(4ac-b^2)(ax^2+bx+c)} + \\ + \frac{1}{2a^2}\ln|ax^2+bx+c| + \frac{b(6ac-b^2)}{2a^2(b^2-4ac)}\int\frac{dx}{ax^2+bx+c}.$$

## 2.11. Интегралы вида $\displaystyle\int\frac{dx}{x^m(ax^2+bx+c)^n}$.

**1.** $$\int \frac{dx}{x^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^2+bx+c)^{n-1}} - \\ - \frac{(m+n-2)b}{(m-1)c}\int\frac{dx}{x^{m-1}(ax^2+bx+c)^n} - \\ - \frac{(m+2n-3)a}{(m-1)c}\int\frac{dx}{x^{m-2}(ax^2+bx+c)^n}.$$

**2.** $$\int \frac{dx}{x(ax^2+bx+c)^n} = \frac{1}{2(n-1)c(ax^2+bx+c)^{n-1}} - \\ - \frac{b}{2c}\int\frac{dx}{(ax^2+bx+c)^n} + \frac{1}{c}\int\frac{dx}{x(ax^2+bx+c)^{n-1}}.$$

**3.** $$\int \frac{dx}{x(ax^2+bx+c)} = \frac{1}{2c}\ln\frac{x^2}{|ax^2+bx+c|} - \\ - \frac{b}{c\sqrt{|b^2-4ac|}} \begin{cases} \dfrac{1}{2}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & [b^2>4ac], \\ \operatorname{arctg}\dfrac{2ax+b}{\sqrt{|4ac-b^2|}} & [b^2<4ac]. \end{cases}$$

**4.** $$\int \frac{dx}{x^2(ax^2+bx+c)} = -\frac{1}{cx} - \frac{b}{2c^2}\ln\frac{x^2}{|ax^2+bx+c|} + \frac{b^2-2ac}{2c^2}\int\frac{dx}{ax^2+bx+c}.$$

**5.** $$\int \frac{dx}{x^3(ax^2+bx+c)} = \frac{2bx-c}{2c^2x^2} + \frac{b^2-ac}{2c^3}\ln\frac{x^2}{|ax^2+bx+c|} + \frac{b(3ac-b^2)}{2c^3}\int\frac{dx}{ax^2+bx+c}.$$

**6.** $$\int \frac{dx}{x(ax^2+bx+c)^2} = \frac{1}{2c(ax^2+bx+c)}\left[1+\frac{b(b+2ax)}{b^2-4ac}\right] + \frac{1}{2c^2}\ln\frac{x^2}{|ax^2+bx+c|} - \frac{b}{2c^2}\left(1-\frac{2ac}{b^2-4ac}\right)\int\frac{dx}{ax^2+bx+c}.$$

**7.** $$\int \frac{dx}{x^2(ax^2+bx+c)^2} = \frac{(b^2-3ac)(b+2ax)}{c^2(4ac-b^2)(ax^2+bx+c)} - \frac{c+bx}{c^2x(ax^2+bx+c)} - \frac{b}{c^3}\ln\frac{x^2}{|ax^2+bx+c|} + \frac{1}{b^2-4ac}\left(\frac{b^4}{c^3}-\frac{6b^2a}{c^2}+\frac{6a^2}{c}\right)\int\frac{dx}{ax^2+bx+c}.$$

**8.** $$\int \frac{dx}{x^3(ax^2+bx+c)^2} = \frac{3bx-c}{2c^2x^2(ax^2+bx+c)} + \frac{3b^2-2ac}{c^2}\int\frac{dx}{x(ax^2+bx+c)^2} + \frac{9ab}{2c^2}\int\frac{dx}{(ax^2+bx+c)^2}.$$

### 2.12. Интегралы вида $\int (x+d)^{\pm m}(ax^2+bx+c)^n\,dx$.

**1.** $$\int (x+d)^m(ax^2+bx+c)^n\,dx = \frac{1}{(m+2n+1)a}\Big[(x+d)^{m-1}(ax^2+bx+c)^{n+1} + (m+n)(2ad-b)\int (x+d)^{m-1}(ax^2+bx+c)^n\,dx - (m-1)(ad^2-bd+c)\int (x+d)^{m-2}(ax^2+bx+c)^n\,dx\Big].$$

**2.** $$\int \frac{(ax^2+bx+c)^n}{(x+d)^m}\,dx = -\frac{(ax^2+bx+c)^n}{(m-2n-1)(x+d)^{m-1}} - \frac{2n(ad^2-bd+c)}{m-2n-1}\int\frac{(ax^2+bx+c)^{n-1}}{(x+d)^m}\,dx - \frac{n(b-2ad)}{m-2n-1}\int\frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-1}}\,dx.$$

**3.** $$= \frac{-1}{(m-1)(ad^2-bd+c)}\Big[\frac{(ax^2+bx+c)^{n+1}}{(x+d)^{m-1}} + (m-n-2)(b-2ad)\int\frac{(ax^2+bx+c)^n}{(x+d)^{m-1}}\,dx + (m-2n-3)a\int\frac{(ax^2+bx+c)^n}{(x+d)^{m-2}}\,dx\Big].$$

**4.** $$= -\frac{(ax^2+bx+c)^n}{(m-1)(x+d)^{m-1}} + \frac{n(b-2ad)}{m-1}\int\frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-1}}\,dx + \frac{2na}{m-1}\int\frac{(ax^2+bx+c)^{n-1}}{(x+d)^{m-2}}\,dx.$$

**5.** $$\int\frac{(ax^2+bx+c)^n}{x+d}\,dx = (ad^2-bd+c)^n\ln|x+d| + \int(ax+b-ad)\left[\sum_{k=0}^{n-1}(ad^2-bd+c)^{n-k-1}(ax^2+bx+c)^k\right]dx.$$

## 2.13. Интегралы вида $\int\frac{(x+d)^m\,dx}{(ax^2+bx+c)^n}$.

**1.** $$\int\frac{(x+d)^m\,dx}{(ax^2+bx+c)^n} = \frac{(2ad-b)(2ax+b)+4ac-b^2}{2(n-1)(4ac-b^2)(ad^2-bd+c)}\,\frac{(x+d)^{m+1}}{(ax^2+bx+c)^{n-1}} + \frac{2(m-2n+4)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)}\,\frac{(x+d)^{m+1}}{(ax^2+bx+c)^{n-2}} + \left[\frac{4(2n-3)a}{(n-1)(4ac-b^2)} - \frac{2(m-n+2)}{(n-1)(ad^2-bd+c)}\right]\int\frac{(x+d)^m\,dx}{(ax^2+bx+c)^{n-1}} - \frac{2(m-2n+4)(m-2n+5)a}{(n-1)(n-2)(4ac-b^2)(ad^2-bd+c)}\int\frac{(x+d)^m\,dx}{(ax^2+bx+c)^{n-2}}.$$

**2.** $$= \frac{1}{(m-2n+1)a}\left[\frac{(x+d)^{m-1}}{(ax^2+bx+c)^{n-1}} + (m-n)(2ad-b)\times\int\frac{(x+d)^{m-1}\,dx}{(ax^2+bx+c)^n} - (m-1)(ad^2-bd+c)\int\frac{(x+d)^{m-2}\,dx}{(ax^2+bx+c)^n}\right].$$

**3.** $$= \frac{1}{(n-1)(4ac-b^2)}\left[\frac{(2ax+b)(x+d)^m}{(ax^2+bx+c)^{n-1}} - 2(m-2n+3)a\int\frac{(x+d)^m\,dx}{(ax^2+bx+c)^{n-1}} - m(b-2ad)\int\frac{(x+d)^{m-1}\,dx}{(ax^2+bx+c)^{n-1}}\right].$$

**4.** $$\int\frac{(x+d)\,dx}{ax^2+bx+c} = \frac{1}{2a}\ln|ax^2+bx+c| + \frac{2ad-b}{a\sqrt{|b^2-4ac|}}\begin{cases}\operatorname{arctg}\dfrac{2ax+b}{\sqrt{4ac-b^2}} & [b^2<4ac],\\[2ex] \dfrac{1}{2}\ln\left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & [b^2>4ac].\end{cases}$$

**5.** $$\int\frac{(x+d)^2\,dx}{ax^2+bx+c} = \frac{x}{a} + \frac{ad^2-c}{a}\int\frac{dx}{ax^2+bx+c} + \frac{2ad-b}{a}\int\frac{x\,dx}{ax^2+bx+c}.$$

**6.** $$\int\frac{(x+d)\,dx}{(ax^2+bx+c)^2} = \frac{(b-2ad)x+2c-bd}{(b^2-4ac)(ax^2+bx+c)} + \frac{b-2ad}{b^2-4ac}\int\frac{dx}{ax^2+bx+c}.$$

**2.14.** Интегралы вида $\int \frac{dx}{(x+d)^m(ax^2+bx+c)^n}$.

**1.** $$\int \frac{dx}{(x+d)^m(ax^2+bx+c)^n} = -\frac{1}{(m-1)(ad^2-bd+c)} \times \\ \times \left[\frac{1}{(x+d)^{m-1}(ax^2+bx+c)^{n-1}} + (m+n-2)(b-2ad) \times \right. \\ \times \int \frac{dx}{(x+d)^{m-1}(ax^2+bx+c)^n} + \\ \left. + (m+2n-3)a \int \frac{dx}{(x+d)^{m-2}(ax^2+bx+c)^n}\right].$$

**2.** $$= \frac{1}{2(ad^2-bd+c)} \times \\ \times \left[\frac{1}{(n-1)(x+d)^{m-1}(ax^2+bx+c)^{n-1}} + 2(ad-b) \times \right. \\ \left. \times \int \frac{dx}{(x+d)^{m-1}(ax^2+bx+c)^n} + \frac{m+2n-3}{n-1} \int \frac{dx}{(x+d)^m(ax^2+bx+c)^{n-1}}\right].$$

**3.** $$= \frac{1}{(m+n-1)(2ad-b)} \left[\frac{1}{(x+d)^m(ax^2+bx+c)^{n-1}} + \right. \\ \left. + (m+2n-2)a \int \frac{dx}{(x+d)^{m-1}(ax^2+bx+c)^n}\right] \qquad [ad^2-bd+c=0].$$

**4.** $$\int \frac{dx}{(x+d)(ax^2+bx+c)^n} = \frac{1}{ad^2-bd+c} \left[\frac{1}{2(n-1)(ax^2+bx+c)^{n-1}} + \right. \\ \left. + \left(ad - \frac{b}{2}\right) \int \frac{dx}{(ax^2+bx+c)^n} + \int \frac{dx}{(x+d)(ax^2+bx+c)^{n-1}}\right].$$

**5.** $$\int \frac{dx}{(x+d)(ax^2+bx+c)} = \frac{1}{2(ad^2-bd+c)} \ln \frac{(x+a)^2}{|ax^2+bx+c|} + \\ + \frac{2ad-b}{2(ad^2-bd+c)\sqrt{|b^2-4ac|}} \begin{cases} 2\,\mathrm{arctg}\, \dfrac{2ax+b}{\sqrt{4ac-b^2}} & [b^2<4ac], \\ \ln \left|\dfrac{2ax+b-\sqrt{b^2-4ac}}{2ax+b+\sqrt{b^2-4ac}}\right| & [b^2>4ac]. \end{cases}$$

**6.** $$\int \frac{dx}{(x+d)^2(ax^2+bx+c)} = \\ = -\frac{1}{ad^2-bd+c} \left[\frac{1}{x+d} + \frac{2ad-b}{2(ad^2-bd+c)} \ln \frac{(x+a)^2}{|ax^2+bx+c|} + \right. \\ \left. + \frac{2c(ad^2-bd+c)-2ad+b^2}{2(ad^2-bd+c)^2} \int \frac{dx}{ax^2+bx+c}\right].$$

**2.15.** Интегралы вида $\int \frac{x^m\,dx}{(x^2+a^2)^n}$.

**1.** $$\int \frac{x^m\,dx}{(x^2+a^2)^n} = \frac{x^{m+1}}{2(n-1)a^2(x^2+a^2)^{n-1}} - \frac{m-2n+3}{2(n-1)a^2} \int \frac{x^m\,dx}{(x^2+a^2)^{n-1}}.$$

**2.** $$= \frac{x^{m-1}}{(m-2n+1)(x^2+a^2)^{n-1}} - \frac{(m-1)a^2}{m-2n+1} \int \frac{x^{m-2}\,dx}{(x^2+a^2)^n}.$$

**3.** $$= -\frac{x^{m-1}}{2(n-1)(x^2+a^2)^{n-1}} + \frac{m-1}{2(n-1)}\int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n-1}}.$$

**4.** $$= \int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n-1}} - a^2 \int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n}}.$$

**5.** $$\int \frac{x^{2m}\,dx}{(x^2+a^2)^n} = \frac{x^{2m+1}}{2(n-1)a^2} \times$$
$$\times \left[\frac{1}{(x^2+a^2)^{n-1}} + \sum_{k=1}^{n-2} \frac{(2n-2m-3)(2n-2m-5)\ldots(2n-2m-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)} \times \right.$$
$$\left. \times \frac{1}{a^{2k}(x^2+a^2)^{n-k-1}}\right] +$$
$$+ (-1)^m a^{2m-2n+1} \frac{(2n-2m-3)(2n-2m-5)\ldots(3-2m)(1-2m)}{2^{n-1}(n-1)!} \times$$
$$\times \left[\sum_{k=1}^{m} \frac{(-1)^k}{2k-1}\left(\frac{x}{a}\right)^{2k-1} + \operatorname{arctg}\frac{x}{a}\right].$$

**6.** $$\int \frac{x^{2m+1}\,dx}{(x^2+a^2)^n} = -\frac{1}{2}\sum_{k=0}^{m}(-1)^k C_m^k \frac{a^{2k}}{(n-m+k-1)(x^2+a^2)^{n-m+k-1}}.$$

**7.** $$= \frac{1}{2a^{2(n-m-1)}} \sum_{k=0}^{n-m-2} \frac{(-1)^k}{m+k+1} C_{n-m-2}^k \left(\frac{x^2}{x^2+a^2}\right)^{m+k+1}$$
$$[n \geqslant m+2].$$

**8.** $$\int \frac{x^{2m}\,dx}{x^2+a^2} = \sum_{k=0}^{m-1}(-1)^k \frac{a^{2k}x^{2m-2k-1}}{2m-2k-1} + (-1)^m a^{2m-1}\operatorname{arctg}\frac{x}{a}.$$

**9.** $$\int \frac{x^{2m+1}\,dx}{x^2+a^2} = \frac{1}{2}\sum_{k=0}^{m-1}\frac{(-1)^k}{m-k}a^{2k}x^{2m-2k} + \frac{(-a^2)^m}{2}\ \ln(x^2+a^2).$$

**10.** $$\int \frac{x^{2m}\,dx}{(x^2+a^2)^2} = \frac{x^{2m+1}}{2a^2(x^2+a^2)} - \frac{2m-1}{2}a^{2m-3} \times$$
$$\times \left[\sum_{k=1}^{m}\frac{(-1)^{m+k}}{2k-1}\left(\frac{x}{a}\right)^{2k-1} + (-1)^m \operatorname{arctg}\frac{x}{a}\right].$$

**11.** $$\int \frac{dx}{(x^2+a^2)^n} = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}\int\frac{dx}{(x^2+a^2)^{n-1}}.$$

**12.** $$= \frac{x}{2n-1}\sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)a^{2k}(x^2+a^2)^{n-k}} +$$
$$+ \frac{(2n-3)!!}{2^{n-1}(n-1)!a^{2n-1}}\operatorname{arctg}\frac{x}{a}.$$

**13.** $$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\operatorname{arctg}\frac{x}{a}.$$

**14.** $$\int \frac{dx}{(x^2+a^2)^2} = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\operatorname{arctg}\frac{x}{a}.$$

15. $\displaystyle\int \frac{dx}{(x^2+a^2)^3} = \frac{x}{4a^2(x^2+a^2)^2} + \frac{3x}{8a^4(x^2+a^2)} + \frac{3}{8a^5} \operatorname{arctg} \frac{x}{a}.$

16. $\displaystyle\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2} \ln (x^2+a^2).$

17. $\displaystyle\int \frac{x^2\,dx}{x^2+a^2} = x - a \operatorname{arctg} \frac{x}{a}.$

18. $\displaystyle\int \frac{x^3\,dx}{x^2+a^2} = \frac{x^2}{2} - \frac{a^2}{2} \ln |x^2+a^2|.$

19. $\displaystyle\int \frac{x^4\,dx}{x^2+a^2} = \frac{x^3}{3} - a^2 x + a^3 \operatorname{arctg} \frac{x}{a}.$

20. $\displaystyle\int \frac{x\,dx}{(x^2+a^2)^2} = -\frac{1}{2(x^2+a^2)}.$

21. $\displaystyle\int \frac{x^2\,dx}{(x^2+a^2)^2} = -\frac{x}{2(x^2+a^2)} + \frac{1}{2a} \operatorname{arctg} \frac{x}{a}.$

22. $\displaystyle\int \frac{x^3\,dx}{(x^2+a^2)^2} = \frac{a^2}{2(x^2+a^2)} + \frac{1}{2} \ln (x^2+a^2).$

23. $\displaystyle\int \frac{x^4\,dx}{(x^2+a^2)^2} = x + \frac{a^2 x}{2(x^2+a^2)} - \frac{3a}{2} \operatorname{arctg} \frac{x}{a}.$

**2.16.** Интегралы вида $\displaystyle\int \frac{dx}{x^m(x^2+a^2)^n}$.

1. $$\int \frac{dx}{x^{2m}(x^2+a^2)^n} = -\frac{1}{(2m-1)a^2x^{2m-1}(x^2+a^2)^{n-1}} - \frac{2m+2n-3}{(2m-1)a^2} \int \frac{dx}{x^{2m-2}(x^2+a^2)^n}.$$

2. $$= \frac{1}{(2m+2n-1)(x^2+a^2)^{n-1}} \times \sum_{k=1}^{m} \frac{(-1)^k(2m+2n-1)(2m+2n-3)\ldots(2m+2n-2k+1)}{(2m-1)(2m-3)\ldots(2m-2k+1)a^{2k}} \frac{1}{x^{2m-2k+1}} + (-1)^m \frac{(2n-1)(2n+1)\ldots(2n+2m-3)}{(2m-1)!!a^{2m}} \int \frac{dx}{(x^2+a^2)^n}$$ [см. 2.15.11–12].

3. $$\int \frac{dx}{x^{2m+1}(x^2+a^2)^n} = -\frac{1}{2ma^2x^{2m}(x^2+a^2)^{n-1}} - \frac{m+n-1}{ma^2} \int \frac{dx}{x^{2m-1}(x^2+a^2)^n}.$$

4. $$= \frac{1}{2(-a^2)^{m+n}} \sum_{k=0}^{m+n-1} \frac{(-1)^k}{k-n+1} C_{m+n-1}^k \left(\frac{x^2+a^2}{x^2}\right)^{k-n+1}.$$

5. $$\int \frac{dx}{x(x^2+a^2)^n} = \frac{1}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{1}{a^2} \int \frac{dx}{x(x^2+a^2)^{n-1}}.$$

6. $$= \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{(n-k)a^{2k}(x^2+a^2)^{n-k}} + \frac{1}{2a^{2n}} \ln \frac{x^2}{x^2+a^2}.$$

**7.** $$\int \frac{dx}{x^{2m}(x^2+a^2)} = \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{(2k+1)a^{2(m-k)}x^{2k+1}} + \frac{(-1)^m}{a^{2m+1}} \operatorname{arctg} \frac{x}{a}.$$

**8.** $$\int \frac{dx}{x^{2m+1}(x^2+a^2)} = \frac{1}{2} \sum_{k=0}^{m-1} \frac{(-1)^{k+1}}{(m-k)a^{2(k+1)}x^{2(m-k)}} + \frac{(-1)^m}{2a^{2(m+1)}} \ln \frac{x^2}{x^2+a^2}.$$

**9.** $$\int \frac{dx}{x^{2m}(x^2+a^2)^2} = \sum_{k=0}^{m-1} \frac{(-1)^{m-k}(m-k)}{(2k+1)a^{2m-2k+2}x^{2k+1}} - \\ - \frac{(-1)^{m+1}x}{2a^{2m+2}(x^2+a^2)} + \frac{(-1)^m(2m+1)}{2a^{2m+3}} \operatorname{arctg} \frac{x}{a}.$$

**10.** $$\int \frac{dx}{x(x^2+a^2)} = \frac{1}{2a^2} \ln \frac{x^2}{x^2+a^2}.$$

**11.** $$\int \frac{dx}{x^2(x^2+a^2)} = -\frac{1}{a^2x} \frac{1}{a^3} \operatorname{arctg} \frac{x}{a}.$$

**12.** $$\int \frac{dx}{x^3(x^2+a^2)} = -\frac{1}{2a^2x^2} - \frac{1}{2a^4} \ln \frac{x^2}{x^2+a^2}.$$

**13.** $$\int \frac{dx}{x^4(x^2+a^2)} = -\frac{1}{3a^2x^3} + \frac{1}{a^4x} + \frac{1}{a^5} \operatorname{arctg} \frac{x}{a}.$$

**14.** $$\int \frac{dx}{x(x^2+a^2)^2} = \frac{1}{2a^2(a^2+x^2)} + \frac{1}{2a^4} \ln \frac{x^2}{x^2+a^2}.$$

**15.** $$\int \frac{dx}{x^2(x^2+a^2)^2} = -\frac{1}{a^4x} - \frac{x}{2a^4(x^2+a^2)} - \frac{3}{2a^5} \operatorname{arctg} \frac{x}{a}.$$

**16.** $$\int \frac{dx}{x^3(x^2+a^2)^2} = -\frac{1}{2a^4x^2} - \frac{1}{2a^4(x^2+a^2)} - \frac{1}{a^6} \ln \frac{x^2}{x^2+a^2}.$$

**17.** $$\int \frac{dx}{x^4(x^2+a^2)^2} = -\frac{1}{3a^4x^3} + \frac{2}{a^6x} + \frac{x}{2a^6(x^2+a^2)} + \frac{5}{2a^7} \operatorname{arctg} \frac{x}{a}.$$

**18.** $$\int \frac{dx}{(x+b)(x^2+a^2)} = \frac{1}{a^2+b^2} \left[ \ln|x+b| - \frac{1}{2} \ln(x^2+a^2) + \frac{b}{a} \operatorname{arctg} \frac{x}{a} \right].$$

### 2.17. Интегралы вида $\int \frac{x^m\,dx}{(x^2-a^2)^n}$.

**1.** $$\int \frac{x^m\,dx}{(x^2-a^2)^n} = -\frac{x^{m+1}}{2(n-1)a^2(x^2-a^2)^{n-1}} + \frac{m-2n+3}{2(n-1)a^2} \int \frac{x^m\,dx}{(x^2-a^2)^{n-1}}.$$

**2.** $$= \frac{x^{m-1}}{(m-2n+1)(x^2-a^2)^{n-1}} + \frac{(m-1)a^2}{m-2n+1} \int \frac{x^{m-2}\,dx}{(x^2-a^2)^n}.$$

**3.** $$= -\frac{x^{m-1}}{2(n-1)(x^2-a^2)^{n-1}} + \frac{m-1}{2(n-1)} \int \frac{x^{m-2}\,dx}{(x^2-a^2)^{n-1}}.$$

**4.** $$= \int \frac{x^{m-2}\,dx}{(x^2-a^2)^{n-1}} + a^2 \int \frac{x^{m-2}\,dx}{(x^2-a^2)^n}.$$

**5.** $$\int \frac{x^{2m}\,dx}{(x^2-a^2)^n} = -\frac{x^{2m+1}}{2(n-1)a^2} \left[ \frac{1}{(x^2-a^2)^{n-1}} - \right. \\ - \sum_{k=1}^{n-2} (-1)^k \frac{(2n-2m-3)(2n-2m-5)\ldots(2n-2m-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)} \times$$

$$\times \frac{1}{a^{2k}(a^2-x^2)^{n-k-1}}\Bigg] +$$

$$+ (-1)^{n+1} a^{2m-2n+1} \frac{(2n-2m-3)(2n-2m-5)\ldots(3-2m)(1-2m)}{2^{n-1}(n-1)!} \times$$

$$\times \left[\sum_{k=1}^{m} \frac{1}{2k-1}\left(\frac{x}{a}\right)^{2k-1} - \frac{1}{2}\ln\left|\frac{x+a}{x-a}\right|\right].$$

**6.** $$\int \frac{x^{2m+1}\,dx}{(x^2-a^2)^n} = -\frac{1}{2}\sum_{k=0}^{m} \frac{C_m^k a^{2k}}{(n-m+k-1)(x^2-a^2)^{n-m+k-1}}.$$

**7.** $$= \frac{(-a^2)^{m-n+1}}{2} \sum_{k=0}^{n-m-2} \frac{(-1)^k C_{n-m-2}^k}{m+k+1}\left(\frac{x^2}{x^2-a^2}\right)^{m+k+1}$$

$[n \geqslant m+2]$.

**8.** $$\int \frac{x^{2m}\,dx}{x^2-a^2} = \sum_{k=0}^{m-1} \frac{a^{2k}x^{2m-2k-1}}{2m-2k-1} + \frac{a^{2m-1}}{2}\ln\left|\frac{x-a}{x+a}\right|.$$

**9.** $$\int \frac{x^{2m+1}\,dx}{x^2-a^2} = \frac{1}{2}\sum_{k=0}^{m-1} \frac{a^{2k}x^{2m-2k}}{m-k} + \frac{a^{2m}}{2}\ln|x^2-a^2|.$$

**10.** $$\int \frac{x^{2m}\,dx}{(x^2-a^2)^2} = -\frac{x^{2m+1}}{2a^2(x^2-a^2)} + \frac{2m-1}{2}a^{2m-3} \times$$

$$\times \left[\sum_{k=1}^{m} \frac{1}{2k-1}\left(\frac{x}{a}\right)^{2k-1} - \frac{1}{2}\ln\left|\frac{x+a}{x-a}\right|\right].$$

**11.** $$\int \frac{dx}{(x^2-a^2)^n} = -\frac{x}{2(n-1)a^2(x^2-a^2)^{n-1}} - \frac{2n-3}{2(n-1)a^2}\int \frac{dx}{(x^2-a^2)^{n-1}}.$$

**12.** $$= \frac{x}{2n-1}\sum_{k=1}^{n-1}(-1)^k \frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k(n-1)(n-2)\ldots(n-k)a^{2k}(x^2-a^2)^{n-k}} +$$

$$+ (-1)^n \frac{(2n-3)!!}{2^{n-1}(n-1)!a^{2n-1}}\ln\left|\frac{x+a}{x-a}\right|.$$

**13.** $$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right|.$$

**14.** $$\int \frac{dx}{(x^2-a^2)^2} = -\frac{x}{2a^2(x^2-a^2)} + \frac{1}{4a^3}\ln\left|\frac{x+a}{x-a}\right|.$$

**15.** $$\int \frac{dx}{(x^2-a^2)^3} = -\frac{x}{4a^2(x^2-a^2)^2} + \frac{3x}{8a^4(x^2-a^2)} + \frac{3}{16a^5}\ln\left|\frac{x-a}{x+a}\right|.$$

**16.** $$\int \frac{x\,dx}{x^2-a^2} = \frac{1}{2}\ln|x^2-a^2|.$$

**17.** $$\int \frac{x^2\,dx}{x^2-a^2} = x - \frac{a}{2}\ln\left|\frac{x+a}{x-a}\right|.$$

**18.** $$\int \frac{x^3\,dx}{x^2-a^2} = \frac{x^2}{2} + \frac{a^2}{2}\ln|x^2-a^2|.$$

**19.** $$\int \frac{x^4\,dx}{x^2-a^2} = \frac{x^3}{3} + a^2x - \frac{a^3}{2}\ln\left|\frac{x+a}{x-a}\right|.$$

**20.** $$\int \frac{x\,dx}{(x^2-a^2)^2} = -\frac{1}{2(x^2-a^2)}.$$

**21.** $$\int \frac{x^2\,dx}{(x^2-a^2)^2} = -\frac{x}{2(x^2-a^2)} + \frac{1}{4a}\ln\left|\frac{x-a}{x+a}\right|.$$

**22.** $$\int \frac{x^3\,dx}{(x^2-a^2)^2} = -\frac{a^2}{2(x^2-a^2)} + \frac{1}{2}\ln|x^2-a^2|.$$

**23.** $$\int \frac{x^4\,dx}{(x^2-a^2)^2} = x - \frac{a^2x}{2(x^2-a^2)} - \frac{3a}{4}\ln\left|\frac{x+a}{x-a}\right|.$$

## 2.18. Интегралы вида $\int \frac{dx}{x^m(x^2-a^2)^n}$.

**1.** $$\int \frac{dx}{x^{2m}(x^2-a^2)^n} = \frac{1}{(2m-1)a^2x^{2m-1}(x^2-a^2)^{n-1}} + \frac{2m+2n-3}{(2m-1)a^2}\int \frac{dx}{x^{2m-2}(x^2-a^2)^n}.$$

**2.** $$= -\frac{1}{(2m+2n-1)(x^2-a^2)^{n-1}} \times \sum_{k=1}^{m} \frac{(2m+2n-1)(2m+2n-3)\ldots(2m+2n-2k+1)}{(2m-1)(2m-3)\ldots(2m-2k+1)a^{2k}x^{2m-2k+1}} + \frac{(2n-1)(2n+1)\ldots(2n+2m-3)}{(2m-1)!!a^{2m}}\int \frac{dx}{(x^2-a^2)^n}$$ [см. 2.17.11].

**3.** $$\int \frac{dx}{x^{2m+1}(x^2-a^2)^n} = \frac{1}{2ma^2x^{2m}(x^2-a^2)^{n-1}} + \frac{m+n-1}{ma^2}\int \frac{dx}{x^{2m-1}(x^2-a^2)^n}.$$

**4.** $$= \frac{1}{2a^{2(m+n)}} \sum_{k=0}^{m+n-1} \frac{(-1)^k}{k-n+1} C_{m+n-1}^k \left(\frac{x^2-a^2}{x^2}\right)^{k-n+1}.$$

**5.** $$\int \frac{dx}{x(x^2-a^2)^n} = -\frac{1}{2(n-1)a^2(x^2-a^2)^{n-1}} - \frac{1}{a^2}\int \frac{dx}{x(x^2-a^2)^{n-1}}.$$

**6.** $$= \frac{1}{2}\sum_{k=1}^{n-1} \frac{(-1)^k}{(n-k)a^{2k}(x^2-a^2)^{n-k}} + \frac{(-1)^n}{2a^{2n}}\ln\frac{x^2}{|x^2-a^2|}.$$

**7.** $$\int \frac{dx}{x^{2m}(x^2-a^2)} = \sum_{k=0}^{m-1} \frac{1}{(2k+1)a^{2(m-k)}x^{2k+1}} + \frac{1}{2a^{2m+1}}\ln\left|\frac{x-a}{x+a}\right|.$$

**8.** $$\int \frac{dx}{x^{2m+1}(x^2-a^2)} = \frac{1}{2}\sum_{k=0}^{m-1} \frac{1}{(m-k)a^{2(k+1)}x^{2(m-k)}} - \frac{1}{2a^{2(m+1)}}\ln\frac{x^2}{|x^2-a^2|}.$$

**9.** $$\int \frac{dx}{x^{2m}(x^2-a^2)^2} = -\sum_{k=0}^{m-1} \frac{(m-k)}{(2k+1)a^{2m-2k+2}x^{2k+1}} - \frac{x}{2a^{2m+2}(x^2-a^2)} + \frac{2m+1}{4a^{2m+3}}\ln\left|\frac{x+a}{x-a}\right|.$$

**10.** $$\int \frac{dx}{x(x^2-a^2)} = \frac{1}{2a^2} \ln \frac{|x^2-a^2|}{x^2}.$$

**11.** $$\int \frac{dx}{x^2(x^2-a^2)} = \frac{1}{a^2 x} + \frac{1}{2a^3} \ln \left| \frac{x-a}{x+a} \right|.$$

**12.** $$\int \frac{dx}{x^3(x^2-a^2)} = \frac{1}{2a^2 x^2} + \frac{1}{2a^4} \ln \frac{|x^2-a^2|}{x^2}.$$

**13.** $$\int \frac{dx}{x^4(x^2-a^2)} = \frac{1}{3a^2 x^3} + \frac{1}{a^4 x} + \frac{1}{2a^5} \ln \left| \frac{x-a}{x+a} \right|.$$

**14.** $$\int \frac{dx}{x(x^2-a^2)^2} = -\frac{1}{2a^2(a^2-x^2)} + \frac{1}{2a^4} \ln \frac{x^2}{|x^2-a^2|}.$$

**15.** $$\int \frac{dx}{x^2(x^2-a^2)^2} = -\frac{1}{a^4 x} - \frac{x}{2a^4(x^2-a^2)} - \frac{3}{4a^5} \ln \left| \frac{x-a}{x+a} \right|.$$

**16.** $$\int \frac{dx}{x^3(x^2-a^2)^2} = -\frac{1}{2a^4 x^2} - \frac{1}{2a^4(x^2-a^2)} + \frac{1}{a^6} \ln \frac{x^2}{|x^2-a^2|}.$$

**17.** $$\int \frac{dx}{x^4(x^2-a^2)^2} = -\frac{1}{3a^4 x^3} - \frac{2}{a^6 x} - \frac{x}{2a^6(x^2-a^2)} + \frac{5}{4a^7} \ln \left| \frac{x+a}{x-a} \right|.$$

**18.** $$\int \frac{dx}{(x+b)(x^2-a^2)} =$$
$$= -\frac{1}{a^2-b^2} \ln |x+b| + \frac{1}{2a(a+b)} \ln |x-a| + \frac{1}{2a(a-b)} \ln |x+a|.$$

## 2.19. Интегралы вида $\int \frac{x^{\pm m}\,dx}{(x^3+a^3)^n}$.

**1.** $$\int \frac{x^m\,dx}{(x^3+a^3)^n} = \frac{x^{m+1}}{3(n-1)a^3(x^3+a^3)^{n-1}} - \frac{m-3n+4}{3(n-1)a^3} \int \frac{x^m dx}{(x^3+a^3)^{n-1}}.$$

**2.** $$= \frac{x^{m-2}}{(m-3n+1)(x^3+a^3)^{n-1}} - \frac{(m-2)a^3}{m-3n+1} \int \frac{x^{m-3}\,dx}{(x^3+a^3)^n}.$$

**3.** $$\int \frac{x^m\,dx}{x^3+a^3} = \frac{x^{m-2}}{m-2} - a^3 \int \frac{x^{m-3}\,dx}{x^3+a^3}.$$

**4.** $$\int \frac{dx}{(x^3+a^3)^n} = \frac{x}{3a^3(n-1)(x^3+a^3)^{n-1}} + \frac{3n-4}{3a^3(n-1)} \int \frac{dx}{(x^3+a^3)^{n-1}}.$$

**5.** $$\int \frac{x\,dx}{(x^3+a^3)^n} = \frac{x^2}{3a^3(n-1)(x^3+a^3)^{n-1}} + \frac{3n-5}{3a^3(n-1)} \int \frac{x\,dx}{(x^3+a^3)^{n-1}}.$$

**6.** $$\int \frac{dx}{x^3+a^3} = \frac{1}{6a^2} \ln \frac{(x+a)^2}{x^2-ax+a^2} + \frac{1}{a^2\sqrt{3}} \operatorname{arctg} \frac{2x-a}{a\sqrt{3}}.$$

**7.** $$\int \frac{x\,dx}{x^3+a^3} = \frac{1}{6a} \ln \frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{a\sqrt{3}} \operatorname{arctg} \frac{2x-a}{a\sqrt{3}}.$$

**8.** $$\int \frac{x^2\,dx}{x^3+a^3} = \frac{1}{3} \ln |x^3+a^3|.$$

**9.** $$\int \frac{x^3\,dx}{x^3+a^3} = x + \frac{a}{6} \ln \frac{x^2-ax+a^2}{(x+a)^2} - \frac{a}{\sqrt{3}} \operatorname{arctg} \frac{2x-a}{a\sqrt{3}}.$$

**10.** $\displaystyle\int \frac{dx}{x(x^3+a^3)} = \frac{1}{3a^3}\ln\left|\frac{x^3}{x^3+a^3}\right|.$

**11.** $\displaystyle\int \frac{dx}{x^2(x^3+a^3)} = -\frac{1}{a^3x} + \frac{1}{6a^4}\ln\frac{(x+a)^2}{x^2-ax+a^2} - \frac{1}{a^4\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

**12.** $\displaystyle\int \frac{dx}{x^3(x^3+a^3)} = -\frac{1}{2a^3x^2} - \frac{1}{6a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} - \frac{1}{a^5\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$

**13.** $\displaystyle\int \frac{dx}{(x^3+a^3)^2} =$

$$= \frac{x}{3a^3(x^3+a^3)} + \frac{1}{9a^5}\ln\frac{(x+a)^2}{x^2-ax+a^2} + \frac{2}{3a^5\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$$

**14.** $\displaystyle\int \frac{x\,dx}{(x^3+a^3)^2} =$

$$= \frac{x^2}{3a^3(x^3+a^3)} + \frac{1}{18a^4}\ln\frac{x^2-ax+a^2}{(x+a)^2} + \frac{1}{3a^4\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$$

**15.** $\displaystyle\int \frac{x^2\,dx}{(x^3+a^3)^2} = -\frac{1}{3(x^3+a^3)}.$

**16.** $\displaystyle\int \frac{dx}{x(x^3+a^3)^2} = \frac{1}{3a^3(x^3+a^3)} + \frac{1}{3a^6}\ln\left|\frac{x^3}{x^3+a^3}\right|.$

**17.** $\displaystyle\int \frac{dx}{x^2(x^3+a^3)^2} = \frac{1}{3a^3x(x^3+a^3)} - \frac{4}{3a^6x} +$

$$+ \frac{2}{9a^7}\ln\frac{(x+a)^2}{x^2-ax+a^2} - \frac{4}{3a^7\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$$

**18.** $\displaystyle\int \frac{dx}{x^3(x^3+a^3)^2} =$

$$= -\frac{1}{3a^3x^2(x^3+a^3)} - \frac{5}{6a^2x^2} - \frac{5}{18a^8}\ln\frac{(x+a)^2}{x^2-ax+a^2} - \frac{5}{3a^8\sqrt{3}}\operatorname{arctg}\frac{2x-a}{a\sqrt{3}}.$$

## 2.20. Интегралы вида $\displaystyle\int \frac{x^m\,dx}{(x^4+a^4)^n}$.

**1.** $\displaystyle\int \frac{x^m\,dx}{(x^4+a^4)^n} = \frac{x^{m+1}}{4(n-1)a^4(x^4+a^4)^{n-1}} + \frac{4n-m-5}{4(n-1)a^4}\int \frac{x^m\,dx}{(x^4+a^4)^{n-1}}.$

**2.** $\displaystyle\qquad = \frac{x^{m-3}}{(m-4n+1)(x^4+a^4)^{n-1}} - \frac{(m-3)a^4}{m-4n+1}\int \frac{x^{m-4}\,dx}{(x^4+a^4)^n}.$

**3.** $\displaystyle\int \frac{x^m\,dx}{x^4+a^4} = \frac{x^{m-3}}{m-3} - a^4\int \frac{x^{m-4}\,dx}{x^4+a^4}.$

**4.** $\displaystyle\int \frac{dx}{x^4+a^4} = \frac{1}{4a^3\sqrt{2}}\ln\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2} + \frac{1}{2a^3\sqrt{2}}\operatorname{arctg}\frac{ax\sqrt{2}}{a^2-x^2}.$

**5.** $\displaystyle\int \frac{x\,dx}{x^4+a^4} = \frac{1}{2a^2}\operatorname{arctg}\frac{x^2}{a^2}.$

**6.** $\displaystyle\int \frac{x^2\,dx}{x^4+a^4} = \frac{1}{4a\sqrt{2}}\ln\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2} + \frac{1}{2a\sqrt{2}}\operatorname{arctg}\frac{ax\sqrt{2}}{a^2-x^2}.$

7. $\displaystyle\int \frac{x^3\,dx}{x^4+a^4} = \frac{1}{4}\ln(x^4+a^4).$

8. $\displaystyle\int \frac{x^m\,dx}{(x^4+a^4)^2} = \frac{x^{m+1}}{4a^4(x^4+a^4)} + \frac{3-m}{4a^4}\int \frac{x^m\,dx}{x^4+a^4}.$

9. $\displaystyle\int \frac{dx}{(x^4+a^4)^2} = \frac{x}{4a^4(a^4+x^4)} + \frac{3}{16a^7\sqrt{2}}\ln\frac{x^2+ax\sqrt{2}+a^2}{x^2-ax\sqrt{2}+a^2} + \frac{3}{8a^7\sqrt{2}}\operatorname{arctg}\frac{ax\sqrt{2}}{a^2-x^2}.$

10. $\displaystyle\int \frac{x\,dx}{(x^4+a^4)^2} = \frac{x^2}{4a^4(a^4+x^4)} + \frac{1}{4a^6}\operatorname{arctg}\frac{x^2}{a^2}.$

11. $\displaystyle\int \frac{x^2\,dx}{(x^4+a^4)^2} = \frac{x^3}{4a^4(x^4+a^4)} + \frac{1}{16a^5\sqrt{2}}\ln\frac{x^2-ax\sqrt{2}+a^2}{x^2+ax\sqrt{2}+a^2} + \frac{1}{8a^5\sqrt{2}}\operatorname{arctg}\frac{ax\sqrt{2}}{a^2-x^2}.$

12. $\displaystyle\int \frac{x^3\,dx}{(x^4+a^4)^2} = -\frac{1}{4(x^4+a^4)}.$

**2.21.** Интегралы вида $\displaystyle\int \frac{dx}{x^m(x^4+a^4)^n}$.

1. $\displaystyle\int \frac{dx}{x^m(x^4+a^4)^n} = -\frac{1}{(m-1)a^4x^{m-1}(x^4+a^4)^{n-1}} - \frac{m+4n-5}{(m-1)a^4}\int \frac{dx}{x^{m-4}(x^4+a^4)^n}.$

2. $\displaystyle\qquad = \frac{1}{a^4}\int \frac{dx}{x^m(x^4+a^4)^{n-1}} - \frac{1}{a^4}\int \frac{dx}{x^{m-4}(x^4+a^4)^n}.$

3. $\displaystyle\int \frac{dx}{x(x^4+a^4)} = \frac{1}{4a^4}\ln\frac{x^4}{x^4+a^4}.$

4. $\displaystyle\int \frac{dx}{x^2(x^4+a^4)} = -\frac{1}{a^4x} - \frac{1}{a^4}\int \frac{x^2\,dx}{x^4+a^4}.$

5. $\displaystyle\int \frac{dx}{x^3(x^4+a^4)} = -\frac{1}{2a^4x^2} - \frac{1}{2a^6}\operatorname{arctg}\frac{x^2}{a^2}.$

6. $\displaystyle\int \frac{dx}{x(x^4+a^4)^2} = \frac{1}{4a^4(x^4+a^4)} + \frac{1}{4a^8}\ln\frac{x^4}{x^4+a^4}.$

7. $\displaystyle\int \frac{dx}{x^2(x^4+a^4)^2} = -\frac{1}{a^8x} - \frac{x^3}{4a^8(x^4+a^4)} - \frac{5}{4a^8}\int \frac{x^2\,dx}{x^4+a^4}.$

**2.22.** Интегралы вида $\displaystyle\int \frac{x^m\,dx}{(x^4-a^4)^n}$.

1. $\displaystyle\int \frac{x^m\,dx}{(x^4-a^4)^n} = -\frac{x^{m+1}}{4(n-1)a^4(x^4-a^4)^{n-1}} + \frac{m-4n+5}{4(n-1)a^4}\int \frac{x^m\,dx}{(x^4-a^4)^{n-1}}.$

2. $\displaystyle\qquad = \frac{x^{m-3}}{(m-4n+1)(x^4-a^4)^{n-1}} + \frac{(m-3)a^4}{m-4n+1}\int \frac{x^{m-4}\,dx}{(x^4-a^4)^n}.$

3. $\displaystyle\int \frac{x^m\,dx}{x^4-a^4} = \frac{x^{m-3}}{m-3} + a^4 \int \frac{x^{m-4}\,dx}{x^4-a^4}.$

4. $\displaystyle\int \frac{dx}{x^4-a^4} = -\frac{1}{4a^3} \ln\left|\frac{x+a}{x-a}\right| - \frac{1}{2a^3} \operatorname{arctg}\frac{x}{a}.$

5. $\displaystyle\int \frac{x\,dx}{x^4-a^4} = \frac{1}{4a^2} \ln \frac{|x^2-a^2|}{x^2+a^2}.$

6. $\displaystyle\int \frac{x^2\,dx}{x^4-a^4} = \frac{1}{4a} \ln\left|\frac{x-a}{x+a}\right| + \frac{1}{2a} \operatorname{arctg}\frac{x}{a}.$

7. $\displaystyle\int \frac{x^3\,dx}{x^4-a^4} = \frac{1}{4} \ln|x^4-a^4|.$

8. $\displaystyle\int \frac{x^m\,dx}{(x^4-a^4)^2} = -\frac{x^{m-1}}{4a^4(x^4-a^4)} + \frac{m-3}{4a^4} \int \frac{x^m\,dx}{x^4-a^4}.$

9. $\displaystyle\int \frac{dx}{(x^4-a^4)^2} = -\frac{x}{4a^4(x^4-a^4)} + \frac{3}{16a^7} \ln\left|\frac{x+a}{x-a}\right| + \frac{3}{8a^7} \operatorname{arctg}\frac{x}{a}.$

10. $\displaystyle\int \frac{x\,dx}{(x^4-a^4)^2} = -\frac{x^2}{4a^4(x^4-a^4)} + \frac{1}{8a^6} \ln \frac{x^2+a^2}{|x^2-a^2|}.$

11. $\displaystyle\int \frac{x^2\,dx}{(x^4-a^4)^2} = -\frac{x^3}{4a^4(x^4-a^4)} - \frac{1}{16a^5} \ln\left|\frac{x-a}{x+a}\right| + \frac{1}{8a^5} \operatorname{arctg}\frac{x}{a}.$

12. $\displaystyle\int \frac{x^3\,dx}{(x^4-a^4)^2} = -\frac{1}{4(x^4-a^4)}.$

**2.23.** Интегралы вида $\displaystyle\int \frac{dx}{x^m(x^4-a^4)^n}$.

1. $$\int \frac{dx}{x^m(x^4-a^4)^n} = \frac{1}{(m-1)a^4x^{m-1}(x^4-a^4)^{n-1}} + \frac{m+4n-5}{(m-1)a^4} \int \frac{dx}{x^{m-4}(x^4-a^4)^n}.$$

2. $$= -\frac{1}{a^4} \int \frac{dx}{x^m(x^4-a^4)^{n-1}} - \frac{1}{a^4} \int \frac{dx}{x^{m-4}(x^4-a^4)^n}.$$

3. $\displaystyle\int \frac{dx}{x(x^4-a^4)} = -\frac{1}{4a^4} \ln \frac{x^4}{|x^4-a^4|}.$

4. $\displaystyle\int \frac{dx}{x^2(x^4-a^4)} = \frac{1}{a^4x} + \frac{1}{a^4} \int \frac{x^2\,dx}{x^4-a^4}.$

5. $\displaystyle\int \frac{dx}{x^3(x^4-a^4)} = \frac{1}{2a^4x^2} + \frac{1}{4a^6} \ln \frac{|x^2-a^2|}{x^2+a^2}.$

6. $\displaystyle\int \frac{dx}{x(x^4-a^4)^2} = -\frac{1}{4a^4(x^4-a^4)} + \frac{1}{4a^8} \ln \frac{x^4}{|x^4-a^4|}.$

7. $\displaystyle\int \frac{dx}{x^2(x^4-a^4)^2} = -\frac{1}{a^8x} - \frac{x^3}{4a^8(x^4-a^4)} - \frac{5}{4a^8} \int \frac{x^2\,dx}{x^4-a^4}.$

## 2.24. Интегралы вида $\int \frac{x^{\pm m}\,dx}{(ax^4+bx^2+c)^n}$.

1. $$\int \frac{x^m\,dx}{(ax^4+bx^2+c)^n} = \frac{x^{m-3}}{(m-4n+1)a(ax^4+bx^2+c)^{n-1}} - \\ - \frac{(m-3)c}{(m-4n+1)a}\int \frac{x^{m-4}\,dx}{(ax^4+bx^2+c)^n} - \frac{(m-2n-1)b}{(m-4n+1)a}\int \frac{x^{m-2}\,dx}{(ax^4+bx^2+c)^n}.$$

2. $$= \frac{1}{a}\int \frac{x^{m-4}\,dx}{(ax^4+bx^2+c)^{n-1}} - \frac{c}{a}\int \frac{x^{m-4}\,dx}{(ax^4+bx^2+c)^n} - \\ - \frac{b}{a}\int \frac{x^{m-2}\,dx}{(ax^4+bx^2+c)^n}.$$

3. $$\int \frac{dx}{(ax^4+bx^2+c)^n} = \frac{abx^3+(b^2-2ac)x}{2(n-1)c(b^2-4ac)(ax^4+bx^2+c)^{n-1}} + \\ + \frac{(4n-7)ab}{2(n-1)c(b^2-4ac)}\int \frac{x^2\,dx}{(ax^4+bx^2+c)^{n-1}} + \\ + \frac{2(n-1)(b^2-4ac)+2ac-b^2}{2(n-1)c(b^2-4ac)}\int \frac{dx}{(ax^4+bx^2+c)^{n-1}}.$$

4. $$\int \frac{dx}{x^m(ax^4+bx^2+c)^n} = -\frac{1}{(m-1)cx^{m-1}(ax^4+bx^2+c)^{n-1}} - \\ - \frac{(m+2n-3)b}{(m-1)c}\int \frac{dx}{x^{m-2}(ax^4+bx^2+c)^n} - \\ - \frac{(m+4n-5)a}{(m-1)c}\int \frac{dx}{x^{m-4}(ax^4+bx^2+c)^n}.$$

5. $$\int \frac{dx}{ax^4+bx^2+c} = \frac{2a}{\sqrt{b^2-4ac}} \times \\ \times \left(\int \frac{dx}{2ax^2+b-\sqrt{b^2-4ac}} - \int \frac{dx}{2ax^2+b+\sqrt{b^2-4ac}}\right) \qquad [b^2>4ac].$$

6. $$= \frac{1}{4\sqrt[4]{ac^3}\,\sin\alpha}\left[\sin\frac{\alpha}{2}\ln\frac{x^2+2x\sqrt[4]{c/a}\,\cos(\alpha/2)+\sqrt{c/a}}{x^2-2x\sqrt[4]{c/a}\,\cos(\alpha/2)+\sqrt{c/a}} + \right. \\ \left. + 2\cos\frac{\alpha}{2}\,\operatorname{arctg}\frac{x^2-\sqrt{c/a}}{2x\sqrt[4]{c/a}\,\sin(\alpha/2)}\right] \qquad \left[b^2<4ac;\ \cos\alpha=-\frac{b}{2\sqrt{ac}}\right].$$

7. $$= \frac{1}{4\sqrt{c(2\sqrt{ac}-b)}}\ln\frac{\sqrt{a}\,x^2+\sqrt{2\sqrt{ac}-b}\,x+\sqrt{c}}{\sqrt{a}\,x^2-\sqrt{2\sqrt{ac}-b}\,x+\sqrt{c}} + \\ + \frac{1}{2\sqrt{c(2\sqrt{ac}+b)}}\operatorname{arctg}\frac{\sqrt{a}\,x^2-\sqrt{c}}{x\sqrt{2\sqrt{ac}+b}} \qquad [a,\ c>0;\ b^2<4ac].$$

8. $$\int \frac{dx}{(ax^4+bx^2+c)^2} = \frac{abx^3+(b^2-2ac)x}{2c(b^2-4ac)(ax^4+bx^2+c)} + \\ + \frac{b^2-6ac}{2a(b^2-4ac)}\int \frac{dx}{ax^4+bx^2+c} + \frac{ab}{2c(b^2-4ac)}\int \frac{x^2\,dx}{ax^4+bx^2+c}.$$

9. $$\int \frac{x\,dx}{ax^4+bx^2+c} = \frac{1}{2\sqrt{b^2-4ac}}\ln\left|\frac{2ax^2+b-\sqrt{b^2-4ac}}{2ax^2+b+\sqrt{b^2-4ac}}\right| \qquad [b^2>4ac].$$

**10.** $$= \frac{1}{\sqrt{4ac - b^2}} \operatorname{arctg} \frac{2ax^2 + b}{\sqrt{4ac - b^2}} \qquad [b^2 < 4ac].$$

**11.** $$\int \frac{x^2\,dx}{ax^4 + bx^2 + c} = \frac{b + \sqrt{b^2 - 4ac}}{\sqrt{b^2 - 4ac}} \int \frac{dx}{2ax^2 + b + \sqrt{b^2 - 4ac}} - \\ - \frac{b - \sqrt{b^2 - 4ac}}{\sqrt{b^2 - 4ac}} \int \frac{dx}{2ax^2 + b - \sqrt{b^2 - 4ac}} \qquad [b^2 > 4ac].$$

**12.** $$= \frac{1}{4a[\sqrt{c/(4a)} - b/(4a)]^{1/2}} \times \\ \times \left\{ \int \frac{x\,dx}{x^2 - 2[\sqrt{c/(4a)} - b/(4a)]^{1/2} x + 2\sqrt{c/(4a)}} - \right. \\ \left. - \int \frac{x\,dx}{x^2 + 2[\sqrt{c/(4a)} - b/(4a)]^{1/2} x + 2\sqrt{c/(4a)}} \right\} \qquad [b^2 < 4ac].$$

### **2.25.** Интегралы вида $\int x^{\pm m}(ax^{2k} + bx^k + c)^{\pm n}\,dx$.

**1.** $$\int x^m (ax^{2k} + bx^k + c)^n\,dx = \frac{x^{m+1}(ax^{2k} + bx^k + c)^{n+1}}{(m+1)c} - \\ - \frac{(m + nk + k + 1)b}{(m+1)c} \int x^{m+k}(ax^{2k} + bx^k + c)^n\,dx - \\ - \frac{(m + 2nk + 2k + 1)a}{(m+1)c} \int x^{m+2k}(ax^{2k} + bx^k + c)^n\,dx.$$

**2.** $$= \frac{x^{m+1}(ax^{2k} + bx^k + c)^n}{m+1} - \frac{nkb}{m+1} \int x^{m+k}(ax^{2k} + bx^k + c)^{n-1}\,dx - \\ - \frac{2nka}{m+1} \int x^{m+2k}(ax^{2k} + bx^k + c)^{n-1}\,dx.$$

**3.** $$= \frac{x^{m-2k+1}(ax^{2k} + bx^k + c)^{n+1}}{(m + 2nk + 1)a} - \frac{(m - 2k + 1)c}{(m + 2nk + 1)a} \times \\ \times \int x^{m-2k}(ax^{2k} + bx^k + c)^n\,dx - \frac{(m + nk - k + 1)b}{(m + 2nk + 1)a} \int x^{m-k}(ax^{2k} + bx^k + c)^n\,dx.$$

**4.** $$= \frac{x^{m+1}(ax^{2k} + bx^k + c)^n}{m + 2nk + 1} + \frac{2nkc}{m + 2nk + 1} \times \\ \times \int x^m (ax^{2k} + bx^k + c)^{n-1}\,dx + \frac{nkb}{m + 2nk + 1} \int x^{m+k}(ax^{2k} + bx^k + c)^{n-1}\,dx.$$

**5.** $$\int \frac{x^m\,dx}{(ax^{2k} + bx^k + c)^n} = \frac{x^{m-2k+1}}{(m - 2nk + 1)a(ax^{2k} + bx^k + c)^{n-1}} - \\ - \frac{(m - 2k + 1)c}{(m - 2nk + 1)a} \int \frac{x^{m-2k}\,dx}{(ax^{2k} + bx^k + c)^n} - \\ - \frac{(m - nk - k + 1)b}{(m - 2nk + 1)a} \int \frac{x^{m-k}\,dx}{(ax^{2k} + bx^k + c)^n}.$$

6. $$\int\frac{dx}{x^m(ax^{2k}+bx^k+c)^n}=-\frac{1}{(m-1)cx^{m-1}(ax^{2k}+bx^k+c)^{n-1}}-\\-\frac{(m+nk-k-1)b}{(m-1)c}\int\frac{dx}{x^{m-k}(ax^{2k}+bx^k+c)^n}-\\-\frac{m+2nk-2k-1}{(m-1)c}\int\frac{dx}{x^{m-2k}(ax^{2k}+bx^k+c)^n}.$$

## 2.26. Интегралы вида $\int x^{m+1/2}(ax\pm b)^{\pm n}\,dx$.

1. $$\int x^{m+1/2}(ax\pm b)^n\,dx=2x^{m+3/2}\sum_{k=0}^{n}C_n^k\frac{a^k(\pm b)^{n-k}x^k}{2k+2m+3}.$$

2. $$\int\frac{x^{m+1/2}\,dx}{(ax\pm b)^n}=-\frac{x^{m+1/2}}{(n-1)a(ax\pm b)^{n-1}}+\frac{2m+1}{2(n-1)a}\int\frac{x^{m-1/2}\,dx}{(ax\pm b)^{n-1}}.$$

3. $$=\frac{2}{a^n}\int\frac{t^{2m+2}\,dt}{(t^2\pm b/a)^n}\qquad[t=\sqrt{x}\,;\ \text{см. } 2.15.11\text{–}12,\ 2.17.11\text{–}12].$$

4. $$\int\frac{x^{m+1/2}\,dx}{ax\pm b}=2\sum_{k=0}^{m}\frac{(\mp 1)^k b^k x^{m-k+1/2}}{(2m-2k+1)a^{k+1}}+\left(\mp\frac{b}{a}\right)^{m-1}\int\frac{dx}{x^{1/2}(ax\pm b)}.$$

5. $$\int\frac{x^{1/2}\,dx}{ax+b}=\frac{2x^{1/2}}{a}-2\left(\frac{b}{a^3}\right)^{1/2}\operatorname{arctg}\sqrt{\frac{ax}{b}}\,.$$

6. $$\int\frac{x^{3/2}\,dx}{ax+b}=\frac{2ax^{3/2}-6bx^{1/2}}{3a^2}+2\left(\frac{b^3}{a^5}\right)^{1/2}\operatorname{arctg}\sqrt{\frac{ax}{b}}\,.$$

7. $$\int\frac{x^{1/2}\,dx}{(ax+b)^2}=-\frac{x^{1/2}}{a(ax+b)}+\frac{1}{(a^3b)^{1/2}}\operatorname{arctg}\sqrt{\frac{ax}{b}}\,.$$

8. $$\int\frac{x^{3/2}\,dx}{(ax+b)^2}=\frac{2ax^{3/2}+3bx^{1/2}}{a^2(ax+b)}-3\left(\frac{b}{a^5}\right)^{1/2}\operatorname{arctg}\sqrt{\frac{ax}{b}}\,.$$

9. $$\int\frac{x^{1/2}\,dx}{ax-b}=\frac{2x^{1/2}}{a}+\left(\frac{b}{a^3}\right)^{1/2}\ln\left|\frac{\sqrt{b}-\sqrt{ax}}{\sqrt{b}+\sqrt{ax}}\right|.$$

10. $$\int\frac{x^{3/2}\,dx}{ax-b}=\frac{2ax^{3/2}+6bx^{1/2}}{3a^2}-\left(\frac{b^3}{a^5}\right)^{1/2}\ln\left|\frac{\sqrt{b}+\sqrt{ax}}{\sqrt{b}-\sqrt{ax}}\right|.$$

11. $$\int\frac{x^{1/2}\,dx}{(ax-b)^2}=-\frac{x^{1/2}}{a(ax-b)}+\frac{1}{2(a^3b)^{1/2}}\ln\left|\frac{\sqrt{b}-\sqrt{ax}}{\sqrt{b}+\sqrt{ax}}\right|.$$

12. $$\int\frac{x^{3/2}\,dx}{(ax-b)^2}=\frac{2ax^{3/2}-3bx^{1/2}}{a^2(ax-b)}-\frac{3}{2}\left(\frac{b}{a^3}\right)^{1/2}\ln\left|\frac{\sqrt{b}+\sqrt{ax}}{\sqrt{b}-\sqrt{ax}}\right|.$$

## 2.27. Интегралы вида $\int\frac{(ax\pm b)^{\pm n}}{x^{m+1/2}}\,dx$.

1. $$\int\frac{(ax\pm b)^n}{x^{m+1/2}}\,dx=\frac{2}{x^{m-1/2}}\sum_{k=0}^{n}C_n^k\frac{a^k(\pm b)^{n-k}x^k}{2k-2m+1}.$$

**2.** $$\int \frac{dx}{x^{m+1/2}(ax \pm b)^n} = \mp \frac{2}{(2m-1)bx^{m-1/2}(ax \pm b)^{n-1}} \mp \mp \frac{(2m+2n-3)a}{(2m-1)b} \int \frac{dx}{x^{m-1/2}(ax \pm b)^n} = \frac{2}{a^n} \int \frac{dt}{t^{2m}(t^2 \pm b/a)^n} \qquad [t = \sqrt{x}].$$

**3.** $$\int \frac{dx}{x^{1/2}(ax+b)} = \frac{2}{(ab)^{1/2}} \operatorname{arctg} \sqrt{\frac{ax}{b}}.$$

**4.** $$\int \frac{dx}{x^{3/2}(ax+b)} = -\frac{2}{bx^{1/2}} - 2\left(\frac{a}{b^3}\right)^{1/2} \operatorname{arctg} \sqrt{\frac{ax}{b}}.$$

**5.** $$\int \frac{dx}{x^{1/2}(ax+b)^2} = \frac{x^{1/2}}{b(ax+b)} + \frac{1}{(ab^3)^{1/2}} \operatorname{arctg} \sqrt{\frac{ax}{b}}.$$

**6.** $$\int \frac{dx}{x^{3/2}(ax+b)^2} = -\frac{3ax+2b}{b^2x^{1/2}(ax+b)} - 3\left(\frac{a}{b^5}\right)^{1/2} \operatorname{arctg} \sqrt{\frac{ax}{b}}.$$

**7.** $$\int \frac{dx}{x^{1/2}(ax-b)} = \frac{1}{(ab)^{1/2}} \ln \left| \frac{\sqrt{b} - \sqrt{ax}}{\sqrt{b} + \sqrt{ax}} \right|.$$

**8.** $$\int \frac{dx}{x^{3/2}(ax-b)} = \frac{2}{bx^{1/2}} + \left(\frac{a}{b^3}\right)^{1/2} \ln \left| \frac{\sqrt{b} - \sqrt{ax}}{\sqrt{b} + \sqrt{ax}} \right|.$$

**9.** $$\int \frac{dx}{x^{1/2}(ax-b)^2} = -\frac{x^{1/2}}{b(ax-b)} + \frac{1}{2(ab^3)^{1/2}} \ln \left| \frac{\sqrt{b} + \sqrt{ax}}{\sqrt{b} - \sqrt{ax}} \right|.$$

**10.** $$\int \frac{dx}{x^{3/2}(ax-b)^2} = \frac{2b-3ax}{b^2x^{1/2}(ax-b)} - \frac{3}{2}\left(\frac{a}{b^5}\right)^{1/2} \ln \left| \frac{\sqrt{b} - \sqrt{ax}}{\sqrt{b} + \sqrt{ax}} \right|.$$

### 2.28. Интегралы вида $\int \frac{x^{m+1/2}}{(x^2+a^2)^n}\, dx$.

**1.** $$\int \frac{x^{m+1/2}}{(x^2+a^2)^n}\, dx = \frac{x^{m+3/2}}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{4n-2m-7}{4(n-1)a^2} \int \frac{x^{m+1/2}}{(x^2+a^2)^{n-1}}\, dx.$$

**2.** $$\int \frac{x^{m+1/2}}{x^2+a^2}\, dx = \frac{2}{2m-1}\, x^{m-1/2} - a^2 \int \frac{x^{m-3/2}}{x^2+a^2}\, dx.$$

**3.** $$\int \frac{x^{2m+1/2}}{x^2+a^2}\, dx = 2\sum_{k=1}^{m} \frac{(-a^2)^{k-1}}{4m-4k+3}\, x^{2m-2k+3/2} + (-a^2)^m \int \frac{x^{1/2}}{x^2+a^2}\, dx.$$

**4.** $$\int \frac{x^{2m+3/2}}{x^2+a^2}\, dx = 2\sum_{k=0}^{m-1} \frac{(-a^2)^{k}}{4m-4k+1}\, x^{2m-2k+1/2} + (-a^2)^m \int \frac{x^{3/2}\, dx}{x^2+a^2}.$$

**5.** $$\int \frac{x^{1/2}\, dx}{x^2+a^2} = \frac{1}{2\sqrt{2a}} \ln \left| \frac{x+a-\sqrt{2ax}}{x+a+\sqrt{2ax}} \right| - \frac{1}{\sqrt{2a}} \operatorname{arctg} \frac{\sqrt{2ax}}{x-a}.$$

**6.** $$\int \frac{x^{3/2}\, dx}{x^2+a^2} = 2\sqrt{x} + \frac{\sqrt{2a}}{4} \ln \left| \frac{x+a-\sqrt{2ax}}{x+a+\sqrt{2ax}} \right| + \sqrt{\frac{a}{2}} \operatorname{arctg} \frac{\sqrt{2ax}}{x-a}.$$

**2.29. Интегралы вида** $\int \frac{x^{m+1/2}}{(x^2-a^2)^n}\,dx$.

**1.** $$\int \frac{x^{m+1/2}}{(x^2-a^2)^n}\,dx = -\frac{x^{m+3/2}}{2(n-1)a^2(x^2-a^2)^{n-1}} - \frac{4n-2m-7}{4(n-1)a^2}\int \frac{x^{m+1/2}}{(x^2-a^2)^{n-1}}\,dx.$$

**2.** $$\int \frac{x^{m+1/2}}{x^2-a^2}\,dx = \frac{2}{2m-1}\,x^{m-1/2} + a^2\int \frac{x^{m-3/2}}{x^2-a^2}\,dx.$$

**3.** $$\int \frac{x^{2m+1/2}}{x^2-a^2}\,dx = 2\sum_{k=1}^{m} \frac{a^{2(k-1)}}{4m-4k+3}\,x^{2m-2k+3/2} + a^{2m}\int \frac{x^{1/2}\,dx}{x^2-a^2}.$$

**4.** $$\int \frac{x^{2m+3/2}}{x^2-a^2}\,dx = 2\sum_{k=0}^{m-1} \frac{a^{2k}}{4m-4k+1}\,x^{2m-2k+1/2} + a^{2m}\int \frac{x^{3/2}\,dx}{x^2-a^2}.$$

**5.** $$\int \frac{x^{1/2}\,dx}{x^2-a^2} = \frac{1}{2\sqrt{a}}\ln\left|\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right| + \frac{1}{\sqrt{a}}\operatorname{arctg}\sqrt{\frac{x}{a}}.$$

**6.** $$\int \frac{x^{3/2}\,dx}{x^2-a^2} = 2\sqrt{x} + \frac{\sqrt{a}}{2}\ln\left|\frac{\sqrt{x}-\sqrt{a}}{\sqrt{x}+\sqrt{a}}\right| + \sqrt{a}\operatorname{arctg}\sqrt{\frac{x}{a}}.$$

**2.30. Интегралы вида** $\int x^m(ax+b)^{n+1/2}\,dx$.

**1.** $$\int x^m(ax+b)^{n+1/2}\,dx = \frac{2(ax+b)^{n+3/2}}{a^{m+1}}\sum_{k=0}^{m}\frac{(-1)^k C_m^k b^k}{2m+2n-2k+3}(ax+b)^{m-k}.$$

**2.** $$\int (ax+b)^{n+1/2}\,dx = \frac{2}{(2n+3)a}\,(ax+b)^{n+3/2}.$$

**3.** $$\int x^m(ax+b)^{1/2}\,dx = \frac{2}{(2m+3)a}\,x^m(ax+b)^{3/2} - \frac{2mb}{(2m+3)a}\int x^{m-1}(ax+b)^{1/2}\,dx.$$

**4.** $$= \frac{2(ax+b)^{3/2}}{a^{m+1}}\sum_{k=0}^{m} C_m^k\frac{(-b)^{m-k}}{2k+3}\,(ax+b)^k.$$

**5.** $$\int (ax+b)^{1/2}\,dx = \frac{2}{3a}\,(ax+b)^{3/2}.$$

**6.** $$\int x(ax+b)^{1/2}\,dx = \frac{2(3ax-2d)}{15a^2}(ax+b)^{3/2}.$$

**7.** $$\int x^2(ax+b)^{1/2}\,dx = \frac{2}{a^3}\left[\frac{(ax+b)^3}{7} - \frac{2b(ax+b)}{5} + \frac{b^2}{3}\right](ax+b)^{3/2}.$$

**8.** $$\int x^3(ax+b)^{1/2}\,dx = \frac{2}{a^4}\left[\frac{(ax+b)^3}{9} - \frac{3b(ax+b)^2}{7} + \frac{3b^2(ax+b)}{5} - \frac{b^3}{3}\right](ax+b)^{3/2}.$$

**9.** $$\int (ax+b)^{3/2}\,dx = \frac{2}{5a}\,(ax+b)^{5/2}.$$

**10.** $$\int x(ax+b)^{3/2}\,dx = \frac{2(5ax-2b)}{35a^2}(ax+b)^{5/2}.$$

**11.** $$\int x^2(ax+b)^{3/2}\,dx = \frac{2}{a^3}\left[\frac{(ax+b)^2}{9} - \frac{2b(ax+b)}{7} + \frac{b^2}{5}\right](ax+b)^{5/2}.$$

**12.** $$\int x^3(ax+b)^{3/2}\,dx = \frac{2}{a^4}\left[\frac{(ax+b)^3}{11} - \frac{3b(ax+b)^2}{9} + \frac{3b^2(ax+b)}{7} - \frac{b^3}{5}\right](ax+b)^{5/2}.$$

### 2.31. Интегралы вида $\int \frac{(ax+b)^{n+1/2}}{x^m}\,dx$.

**1.** $$\int \frac{(ax+b)^{n+1/2}}{x^m}\,dx = -\frac{(ax+b)^{n+3/2}}{(m-1)bx^{m-1}} + \frac{(2n-2m+5)a}{2(m-1)b}\int \frac{(ax+b)^{n+1/2}}{x^{m-1}}\,dx.$$

**2.** $$= \frac{2(ax+b)^{n+1/2}}{(2n-2m+3)x^{m-1}} + \frac{(2n+1)b}{2n-2m+3}\int \frac{(ax+b)^{n-1/2}}{x^m}\,dx.$$

**3.** $$\int \frac{(ax+b)^{n+1/2}}{x}\,dx = \frac{2(ax+b)^{n+1/2}}{2n+1} + b\int \frac{(ax+b)^{n-1/2}}{x}\,dx.$$

**4.** $$\int \frac{(ax+b)^{1/2}}{x^m}\,dx = -\frac{(ax+b)^{3/2}}{(m-1)bx^{m-1}} + \frac{(5-2m)a}{2(m-1)b}\int \frac{(ax+b)^{1/2}}{x^{m-1}}\,dx.$$

**5.** $$\int \frac{(ax+b)^{1/2}}{x}\,dx = 2(ax+b)^{1/2} + \begin{cases} b^{1/2}\ln\dfrac{|(ax+b)^{1/2}-b^{1/2}|}{(ax+b)^{1/2}+b^{1/2}} & [b>0], \\ -2(-b)^{1/2}\operatorname{arctg}\dfrac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b<0]. \end{cases}$$

**6.** $$\int \frac{(ax+b)^{1/2}}{x^2}\,dx = -\frac{(ax+b)^{1/2}}{x} + \frac{a}{2}\int \frac{dx}{x(ax+b)^{1/2}}.$$

**7.** $$\int \frac{(ax+b)^{1/2}}{x^3}\,dx = -\frac{ax+2b}{4bx^2}\,(ax+b)^{1/2} - \frac{a^2}{8b}\int \frac{dx}{x(ax+b)^{1/2}}.$$

**8.** $$\int \frac{(ax+b)^{3/2}}{x}\,dx = \frac{2}{3}\,(ax+b)^{3/2} + 2b(ax+b)^{1/2} + b^2\int \frac{dx}{x(ax+b)^{1/2}}.$$

**9.** $$\int \frac{(ax+b)^{3/2}}{x^2}\,dx = -\frac{(ax+b)^{5/2}}{bx} + \frac{3a}{2b}\int \frac{(ax+b)^{3/2}}{x}\,dx.$$

**10.** $$\int \frac{(ax+b)^{3/2}}{x^3}\,dx = -\left(\frac{1}{2bx^2} + \frac{a}{4b^2x}\right)(ax+b)^{5/2} + \frac{3a^2}{2b^2}\int \frac{(ax+b)^{3/2}}{x}\,dx.$$

**11.** $$\int \frac{(ax+b)^{1/2}}{x-c}\,dx = 2(ax+b)^{1/2} + \\ + |ac+b|^{1/2} \begin{cases} \ln\left|\dfrac{(ax+b)^{1/2}-(ac+b)^{1/2}}{(ax+b)^{1/2}+(ac+b)^{1/2}}\right| & [ac+b>0], \\ 2\operatorname{arctg}\left(\dfrac{|ac+b|}{ax+b}\right)^{1/2} & [ac+b<0]. \end{cases}$$

### 2.32. Интегралы вида $\int \frac{x^m\,dx}{(ax+b)^{n+1/2}}$.

**1.** $$\int \frac{x^m\,dx}{(ax+b)^{n+1/2}} = \frac{2}{a^{m+1}(ax+b)^{n-1/2}} \sum_{k=0}^{m} \frac{(-1)^k C_m^k b^k}{2m-2n-2k+1}(ax+b)^{m-k}.$$

**2.** $$\int \frac{dx}{(ax+b)^{n+1/2}} = -\frac{2}{(2n-1)a(ax+b)^{n-1/2}}.$$

**3.** $$\int \frac{x^m\,dx}{(ax+b)^{1/2}} = \frac{2}{(2m+1)a}\,x^m(ax+b)^{1/2} - \frac{2mb}{(2m+1)a}\int \frac{x^{m-1}dx}{(ax+b)^{1/2}}.$$

**4.** $$= \frac{2(ax+b)^{1/2}}{a^{m+1}} \sum_{k=0}^{m} \frac{(-1)^k C_m^k b^k}{2m-2k+1}(ax+b)^{m-k}.$$

**5.** $$\int \frac{dx}{(ax+b)^{1/2}} = \frac{2}{a}\,(ax+b)^{1/2}.$$

**6.** $$\int \frac{x\,dx}{(ax+b)^{1/2}} = \frac{2(ax-2b)}{3a^2}\,(ax+b)^{1/2}.$$

**7.** $$\int \frac{x^2\,dx}{(ax+b)^{1/2}} = \frac{2}{a^3}\left[\frac{(ax+b)^2}{5} - \frac{2b(ax+b)}{3} + b^2\right](ax+b)^{1/2}.$$

**8.** $$\int \frac{x^3\,dx}{(ax+b)^{1/2}} = \frac{2}{a^4}\left[\frac{(ax+b)^3}{7} - \frac{3b(ax+b)^2}{5} + ab^2x\right](ax+b)^{1/2}.$$

**9.** $$\int \frac{dx}{(ax+b)^{3/2}} = -\frac{2}{a(ax+b)^{1/2}}.$$

**10.** $$\int \frac{x\,dx}{(ax+b)^{3/2}} = \frac{2(ax+2b)}{a^2(ax+b)^{1/2}}.$$

**11.** $$\int \frac{x^2\,dx}{(ax+b)^{3/2}} = \frac{2}{a^3}\left[\frac{(ax+b)^2}{3} - 2b(ax+b) - b^2\right]\frac{1}{(ax+b)^{1/2}}.$$

**12.** $$\int \frac{x^3\,dx}{(ax+b)^{3/2}} = \\ = \frac{2}{a^4}\left[\frac{(ax+b)^3}{5} - b(ax+b)^2 + 3b^2(ax+b) + b^3\right]\frac{1}{(ax+b)^{1/2}}.$$

### 2.33. Интегралы вида $\int \frac{dx}{x^m(ax+b)^{n+1/2}}$.

**1.** $$\int \frac{dx}{x^m(ax+b)^{n+1/2}} = -\frac{1}{(m-1)x^{m-1}(ax+b)^{n+1/2}} - \\ - \frac{(2n+1)a}{2(m-1)}\int \frac{dx}{x^{m-1}(ax+b)^{n+3/2}}.$$

**2.** $$= -\frac{2}{(2n-1)ax^m(ax+b)^{n-1/2}} - \frac{2m}{(2n-1)a}\int \frac{dx}{x^{m+1}(ax+b)^{n-1/2}}.$$

**3.** $$= 2a^{m-1}\int \frac{dt}{t^{2n}(t^2-b)^m} \qquad \left[t = \sqrt{ax+b};\ \text{см. } 2.16,\ 2.18\right].$$

**4.** $$\int \frac{dx}{x(ax+b)^{n+1/2}} = \frac{2}{(2n-1)b(ax+b)^{n-1/2}} + \frac{1}{b}\int \frac{dx}{x(ax+b)^{n-1/2}}.$$

**5.** $$\int \frac{dx}{x^m(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{(m-1)bx^{m-1}} - \frac{(2m-3)a}{2(m-1)b}\int \frac{dx}{x^{m-1}(ax+b)^{1/2}}.$$

**6.** $$\int \frac{dx}{x(ax+b)^{1/2}} = \begin{cases} \dfrac{1}{b^{1/2}} \ln \left| \dfrac{(ax+b)^{1/2} - b^{1/2}}{(ax+b)^{1/2} + b^{1/2}} \right| & [b > 0], \\[2ex] \dfrac{2}{(-b)^{1/2}} \operatorname{arctg} \dfrac{(ax+b)^{1/2}}{(-b)^{1/2}} & [b < 0]. \end{cases}$$

**7.** $$\int \frac{dx}{x^2(ax+b)^{1/2}} = -\frac{(ax+b)^{1/2}}{bx} - \frac{a}{2b}\int \frac{dx}{x(ax+b)^{1/2}}.$$

**8.** $$\int \frac{dx}{x^3(ax+b)^{1/2}} = \frac{3ax-2b}{4b^2x^2}(ax+b)^{1/2} + \frac{3a^2}{8b^2}\int \frac{dx}{x(ax+b)^{1/2}}.$$

**9.** $$\int \frac{dx}{x(ax+b)^{3/2}} = \frac{2}{b(ax+b)^{1/2}} + \frac{1}{b}\int \frac{dx}{x(ax+b)^{1/2}}.$$

**10.** $$\int \frac{dx}{x^2(ax+b)^{3/2}} = \frac{-3ax-b}{b^2x(ax+b)^{1/2}} - \frac{3a}{2b^2}\int \frac{dx}{x(ax+b)^{1/2}}.$$

**11.** $$\int \frac{dx}{x^3(ax+b)^{3/2}} = \left(-\frac{1}{2bx^2} + \frac{5a}{4b^2x} + \frac{15a^2}{4b^3}\right)\frac{1}{(ax+b)^{1/2}} + \\ + \frac{15a^2}{8b^3}\int \frac{dx}{x(ax+b)^{1/2}}.$$

**12.** $$\int \frac{(ax+b)^{1/2}+c}{(ax+b)^{1/2}+d}\,dx = x + 2\frac{c-d}{a}\left[(ax+b)^{1/2} - d\ln\left|(ax+b)^{1/2}+d\right|\right].$$

## **2.34.** Интегралы вида $\int (ax+b)^{\pm m+1/2}(cx+d)^{\pm n+1/2}\,dx$.

**1.** $$\int (ax+b)^{m+1/2}(cx+d)^{n+1/2}\,dx = \frac{(ax+b)^{m+1/2}(cx+d)^{n+3/2}}{(m+n+2)c} - \\ - \frac{(2m+1)(ad-bc)}{2(m+n+2)c}\int (ax+b)^{m-1/2}(cx+d)^{n+1/2}\,dx.$$

**2.** $$= (cx+d)^{n+3/2} \times \\ \times \sum_{k=0}^{m} \frac{(-1)^k(2m+1)(2m-1)\ldots(2m-2k+3)(ad-bc)^k}{(m+n+2)(m+n+1)\ldots(m+n-k+2)c^{k+1}} \times \\ \times (ax+b)^{m-k+1/2} + \frac{m!!}{(-2c)^{m+1}}\sqrt{ax+b} \times \\ \times \sum_{k=0}^{n} \frac{(2n+1)(2n-1)\ldots(2n-2k+3)(ad-bc)^{m+k+1}}{2^k(m+n+2)(m+n+1)\ldots(n-k+1)a^{k+1}} \times \\ \times (cx+d)^{n-k+1/2} + \frac{m!!n!!(ad-bc)^{m+n+2}}{(m+n+2)!(-2c)^{m+1}(2a)^{n+1}}\int \frac{dx}{\sqrt{(ax+b)(cx+d)}}.$$

**3.** $$\int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n+1/2}}\,dx = \frac{1}{(m-n+1)c}\,\frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}} - \frac{(2m+1)(ad-bc)}{2(m-n+1)c}\int \frac{(ax+b)^{m-1/2}}{(cx+d)^{n+1/2}}\,dx.$$

**4.** $$= \frac{2}{(2n-1)(ad-bc)}\,\frac{(ax+b)^{m+3/2}}{(cx+d)^{n-1/2}} + \frac{2(n-m-2)a}{(2n-1)(ad-bc)}\int \frac{(ax+b)^{m+1/2}}{(cx+d)^{n-1/2}}\,dx.$$

**5.** $$= (ax+b)^{m+3/2} \times \sum_{k=0}^{n-m-2} \frac{(n-m-2)(n-m-3)\ldots(n-m-k-1)2^{k+1}a^k}{(2n-1)(2n-3)\ldots(2n-2k-1)(ad-bc)^{k+1}} \times \frac{1}{(cx+d)^{n-k-1/2}} \qquad [n \geqslant m+2].$$

**6.** $$\int \frac{dx}{(ax+b)^{m+1/2}(cx+d)^{n+1/2}} = -\frac{2}{(2m-1)(ad-bc)(ax+b)^{m-1/2}(cx+d)^{n-1/2}} - \frac{2(m+n-1)}{(2m-1)(ad-bc)}\int \frac{dx}{(ax+b)^{m-1/2}(cx+d)^{n+1/2}}.$$

**7.** $$= \frac{1}{(cx+d)^{n-1/2}} \times \sum_{k=0}^{m-1} \frac{(m+n-1)(m+n-2)\ldots(m+n-k)(-2)^{k+1}c^k}{(2m-1)(2m-3)\ldots(2m-2k-1)(ad-bc)^{k+1}} \times \frac{1}{(ax+b)^{m-k-1/2}} + \frac{(-2c)^m}{(m-1)!!}(ax+b)^{1/2} \times \sum_{k=0}^{n-1} \frac{(m+n-1)(m+n-2)\ldots(n-k)2^{k+1}a^k}{(2n-1)(2n-3)\ldots(2n-2k-1)(ad-bc)^{n+k+1}}\,\frac{1}{(cx+d)^{n-k-1/2}}.$$

**8.** $$\int \frac{dx}{\sqrt{(ax+b)(cx+d)}} = \frac{2}{\sqrt{ac}}\ln\left(\sqrt{c(ax+b)} + \sqrt{a(cx+d)}\right) \qquad [ac>0;\ \ x>\max(-b/a,\ -d/c)].$$

**9.** $$= -\frac{2}{\sqrt{ac}}\ln\left(\sqrt{-c(ax+b)} + \sqrt{-a(cx+d)}\right) \qquad [ac>0;\ \ x<\max(-b/a,\ -d/c)].$$

**10.** $$= -\frac{1}{\sqrt{-ac}}\ \arcsin\frac{c(ax+b)+a(cx+d)}{|ad-bc|} \qquad [ac<0].$$

**11.** $$\int \frac{dx}{(x-\alpha)\sqrt{(ax+b)(cx+d)}} = \frac{1}{\sqrt{(a\alpha+b)(c\alpha+d)}}\ln\frac{\left[\sqrt{(c\alpha+d)(ax+b)} - \sqrt{(a\alpha+b)(cx+d)}\right]^2}{|x-\alpha|}$$

$$[(a\alpha+b)(c\alpha+d)>0;\ \ (c\alpha+d)(ax+b)>0;\ \ (a\alpha+b)(cx+d)>0].$$

**12.** $$= \frac{1}{\sqrt{(a\alpha+b)(c\alpha+d)}} \times$$
$$\times \ln \frac{\left[\sqrt{-(c\alpha+d)(ax+b)} - \sqrt{-(a\alpha+b)(cx+d)}\right]^2}{|x-\alpha|}$$
$$[(a\alpha+b)(c\alpha+d) > 0,\ (c\alpha+d)(ax+b) < 0,\ (a\alpha+b)(cx+d) < 0].$$

**13.** $$= \frac{1}{\sqrt{-(a\alpha+b)(c\alpha+d)}} \times$$
$$\times \arcsin \frac{(c\alpha+d)(ax+b)+(a\alpha+b)(cx+d)}{|(ad-bc)(x-\alpha)|} \qquad [(a\alpha+b)(c\alpha+d) < 0].$$

**2.35.** Интегралы вида $\int x^p (ax+b)^{\pm 1/n}\,dx$.

**1.** $$\int f(x,\ \sqrt[n]{ax+b})\,dx = \frac{n}{a}\int f\left(\frac{t^n-b}{a},\ t\right)t^{n-1}\,dt \qquad \left[t = \sqrt[n]{ax+b}\right].$$

**2.** $$\int x^p (ax+b)^{1/n}\,dx =$$
$$= \frac{n}{(np+n+1)a}x^p(ax+b)^{1+1/n} - \frac{bnp}{(np+n+1)a}\int x^{p-1}(ax+b)^{1/n}\,dx.$$

**3.** $$\int x^m (ax+b)^{1/n}\,dx = \frac{1}{a^{m+1}}(ax+b)^{1+1/n}\sum_{k=0}^{m}\frac{C_m^k(-b)^k}{m-k+1+1/n}(ax+b)^{m-k}.$$

**4.** $$\int (ax+b)^{1/n}\,dx = \frac{n}{(n+1)a}(ax+b)^{1+1/n}.$$

**5.** $$\int \frac{(ax+b)^{1/n}}{x^m}\,dx = -\frac{(ax+b)^{1+1/n}}{(m-1)bx^{m-1}} - \frac{(mn-2n-1)a}{(m-1)nb}\int \frac{(ax+b)^{1/n}}{x^{m-1}}\,dx.$$

**6.** $$\int \frac{(ax+b)^{1/n}}{x}\,dx = n(ax+b)^{1/n} + nb^{1/n}\int \frac{dt}{t^n-1}$$
$$\left[t = (ax/b+1)^{1/n};\ \text{см. 2.3}\right].$$

**7.** $$\int \frac{x^m\,dx}{(ax+b)^{1/n}} = \frac{n}{(mn+n-1)a}x^m(ax+b)^{1-1/n} -$$
$$- \frac{mnb}{(mn+n-1)a}\int \frac{x^{m-1}\,dx}{(ax+b)^{1/n}}.$$

**8.** $$= \frac{1}{a^{m+1}}(ax+b)^{1-1/n}\sum_{k=0}^{m}\frac{C_m^k(-b)^k}{m-k+1-1/n}(ax+b)^{m-k}.$$

**9.** $$\int \frac{dx}{(ax+b)^{1/n}} = \frac{n}{(n-1)a}(ax+b)^{1-1/n}.$$

**2.36.** Интегралы вида $\int (x-\alpha)^{\pm m}(ax+b)^{p/n}\,dx$.

**1.** $$\int x^m (ax+b)^{p/n}\,dx =$$
$$= \frac{n}{(mn+n+p)a}x^m(ax+b)^{1+p/n} - \frac{mnb}{(mn+n+p)a}\int x^{m-1}(ax+b)^{p/n}\,dx.$$

**2.** $$= \frac{n}{a}(ax+b)^{1+p/n} \times \sum_{k=0}^{m} \frac{m(m-1)\dots(m-k+1)}{(mn+p+n)(mn+p)\dots(mn+n+p-kn)}\left(-\frac{bn}{a}\right)^k x^{m-k}.$$

**3.** $$= \frac{n}{a^{m+1}}(ax+b)^{1+p/n}\sum_{k=0}^{m}\frac{C_m^k(-b)^k}{(m-k+1)n+p}(ax+b)^{m-k}.$$

**4.** $$\int (ax+b)^{p/n}\,dx = \frac{n}{(n+p)a}(ax+b)^{1+p/n}.$$

**5.** $$\int \frac{(ax+b)^{p/n}}{(x-\alpha)^m}\,dx = -\frac{1}{(m-1)(a\alpha+b)}\frac{(ax+b)^{1+p/n}}{(x-\alpha)^{m-1}} - \frac{[(m-2)n-p]a}{(m-1)n(a\alpha+b)}\int\frac{(ax+b)^{p/n}}{(x-\alpha)^{m-1}}\,dx.$$

**6.** $$= (ax+b)^{1+p/n} \times \sum_{k=1}^{n-1}(-1)^k\frac{(m-p-2n)(mn-p-3n)\dots(mn-p-nk)a^{k-1}}{(m-1)(m-2)\dots(m-k)n^{k-1}(a\alpha+b)^k} \times \frac{1}{(x-\alpha)^{m-k}} - \frac{p(p+1)\dots(p+m-1)a^{m-1}}{n^{m-1}(m-1)!(a\alpha+b)^{m-1}}\int\frac{(ax+b)^{p/n}}{x-\alpha}\,dx.$$

**7.** $$= -\frac{na^{m-1}}{(m-1)n-p}(ax+b)^{1-m+p/n} \qquad [a\alpha+b=0].$$

**8.** $$\int\frac{(ax+b)^{p/n}}{x-\alpha}\,dx = n(a\alpha+b)^{p/n}\int\frac{y^{n+p-1}}{y^n-1}\,dy \qquad \left[y=\sqrt[n]{\frac{ax+b}{a\alpha+b}};\ \text{см. } 2.3\right].$$

## **2.37.** Интегралы вида $\int (x-\alpha)^{\pm m}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx$.

**1.** $$\int f\left(x,\ \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx = \frac{n(ad-bc)}{ac}\int f\left[\frac{-adt^n+bc}{ac(t^n-1)},\ \sqrt[n]{\frac{a}{c}}\,t\right]\frac{t^{n-1}}{(t^n-1)^2}\,dt \qquad \left[t=\sqrt{\frac{c(ax+b)}{a(cx+d)}}\right].$$

**2.** $$\int\left(\frac{ax+b}{cx+d}\right)^{p/n}dx = \frac{ax+b}{a}\left(\frac{ax+b}{cx+d}\right)^{p/n} + \frac{p(ad-bc)}{ac}\left(\frac{a}{c}\right)^{p/n}\int\frac{t^{n+p-1}}{t^n-1}\,dt \qquad \left[t=\sqrt[n]{\frac{c(ax+b)}{a(cx+d)}};\ \text{см. } 2.3\right].$$

**3.** $$\int x^m\left(\frac{ax+b}{cx+d}\right)^{p/n}dx = \frac{(ax+b)(cx+d)x^{m-1}}{(m+1)ac}\left(\frac{ax+b}{cx+d}\right)^{p/n} - \frac{(ad-bc)p+mn(ad+bc)}{(m+1)nac}\int x^{m-1}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx - \frac{(m-1)bd}{(m+1)ac}\int x^{m-2}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx.$$

**4.** $$\int x\left(\frac{ax+b}{cx+d}\right)^{p/n}dx = \frac{(ax+b)(cx+d)}{2ac}\left(\frac{ax+b}{cx+d}\right)^{p/n} - \frac{p(ad-bc)+n(ad+bc)}{2nac}\int\left(\frac{ax+b}{cx+d}\right)^{p/n}dx.$$

**5.** $$\int\frac{1}{(x-\alpha)^{m+1}}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx = A\frac{(ax+b)(cx+d)}{(x-\alpha)^m}\left(\frac{ax+b}{cx+d}\right)^{p/n} + B\int\frac{1}{(x-\alpha)^m}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx + D\int\frac{1}{(x-\alpha)^{m-1}}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx,$$

$$A = -\frac{1}{m(a\alpha+b)(c\alpha+d)},\quad B = \frac{-n(m-1)(ad+bc+2ac\alpha)+p(ad-bc)}{mn(a\alpha+b)(c\alpha+d)},$$

$$D = \frac{-(m-2)ac}{m(a\alpha+b)(c\alpha+d)}.$$

**6.** $$\int\frac{1}{x-\alpha}\left(\frac{ax+b}{cx+d}\right)^{p/n}dx = n\left(\frac{a\alpha+b}{c\alpha+d}\right)^{p/n}\int\frac{t^{p-1}}{t^n-1}\,dt - n\left(\frac{a}{c}\right)^{p/n}\int\frac{s^{p-1}}{s^n-1}\,ds$$

$$\left[t = \sqrt[n]{\frac{(c\alpha+a)(ax+b)}{(a\alpha+b)(cx+d)}},\ s = \sqrt[n]{\frac{c(ax+b)}{a(cx+d)}};\ \text{см. } 2.3\right].$$

## 2.38. Интегралы вида $\int\frac{x^{p/n}\,dx}{(x^2+a^2)^m}$.

**1.** $$\int\frac{x^{p/n}\,dx}{(x^2+a^2)^m} = \frac{x^{p/n+1}}{2(m-1)a^2(x^2+a^2)^{m-1}} + \frac{(2m-3)n-p}{2(m-1)na^2}\int\frac{x^{p/n}\,dx}{(x^2+a^2)^{m-1}}.$$

**2.** $$= x^{p/n+1}\times\sum_{k=1}^{m-1}\frac{(2mn-3n-p)(2mn-5n-p)\ldots(2mn-2kn+n-p)}{(m-1)(m-2)\ldots(m-k)n^{k-1}(2a^2)^k}\times\frac{1}{(x^2+a^2)^{m-k}} + \frac{(n-p)(3n-p)\ldots(2mn-3n-p)}{(m-1)!(2a^2n)^{m-1}}\int\frac{x^{p/n}\,dx}{x^2+a^2}.$$

**3.** $$\int\frac{x^{p/n}\,dx}{x^2+a^2} = \frac{n}{p-n}x^{p/n-1} - a^2\int\frac{x^{p/n-2}\,dx}{x^2+a^2}.$$

**4.** $$\int\frac{dx}{x^{p/n}(x^2+a^2)} = \frac{nx^{1-p/n}}{(n-p)a^2} - \frac{1}{a^2}\int\frac{dx}{x^{p/n-2}(x^2+a^2)}.$$

**5.** $$\int\frac{x^{p/n}\,dx}{x^2+a^2} = \frac{1}{a^{1-p/n}}\times\sum_{k=0}^{n-1}\left(\sin\frac{4k+1}{2n}p\pi\ln\sqrt{x^{2/n}-2a^{1/n}x^{1/n}\cos\frac{4k+1}{2n}\pi+a^{2/n}} + \cos\frac{4k+1}{2n}p\pi\operatorname{arctg}\frac{x^{1/n}-a^{1/n}\cos(4k+1)\pi/2n}{a^{1/n}\sin(4k+1)\pi/2n}\right)$$

$$[p=1,\,2,\,\ldots,\,n-1;\ a>0].$$

**2.39.** Интегралы вида $\int \frac{x^{p/n}\,dx}{(x^2-a^2)^m}$.

**1.** $$\int \frac{x^{p/n}\,dx}{(x^2-a^2)^m} = -\frac{x^{p/n+1}}{2(m-1)a^2(x^2-a^2)^{m-1}} - \frac{(2m-3)n-p}{2(m-1)na^2}\int \frac{x^{p/n}\,dx}{(x^2-a^2)^{m-1}}.$$

**2.** $$= x^{p/n+1} \times \sum_{k=1}^{m-1}(-1)^k \frac{(2mn-3n-p)(2mn-5n-p)\ldots(2mn-2nk+n-p)}{(m-1)(m-2)\ldots(m-k)n^{k-1}(2a^2)^k(x^2-a^2)^{m-k}} - (-1)^m \frac{(n-p)(3n-p)\ldots(2mn-3n-p)}{(m-1)!(2a^2n)^{m-1}}\int \frac{x^{p/n}\,dx}{x^2-a^2}.$$

**3.** $$\int \frac{x^{p/n}\,dx}{x^2-a^2} = \frac{n}{p-n}\,x^{p/n-1} + a^2\int \frac{x^{p/n-2}\,dx}{x^2-a^2}.$$

**4.** $$\int \frac{dx}{x^{p/n}(x^2-a^2)} = -\frac{nx^{1-p/n}}{(n-p)a^2} + \frac{1}{a^2}\int \frac{dx}{x^{p/n-2}(x^2-a^2)}.$$

**5.** $$\int \frac{x^{p/n}}{x^2-a^2}\,dx = \frac{na^{p/n-1}}{2}\int \frac{t^{n+p-1}}{t^n-1}\,dt - \frac{na^{p/n-1}}{2}\int \frac{t^{n+p-1}\,dt}{t^n+1}$$
$$\left[t=(x/a)^{1/n},\ a>0;\ \text{см. } 2.3\right].$$

**2.40.** Интегралы вида $\int x^m(x^2+a^2)^{n+1/2}\,dx$.

**1.** $$\int x^m(x^2+a^2)^{n+1/2}\,dx = \frac{x^{m-1}(x^2+a^2)^{n+3/2}}{m+2n+2} - \frac{(m-1)a^2}{m+2n+2}\int x^{m-2}(x^2+a^2)^{n+1/2}\,dx.$$

**2.** $$\int x^{2m}(x^2+a^2)^{n+1/2}\,dx = \frac{(x^2+a^2)^{n+3/2}}{2m+2n+2}\Bigg[x^{2m-1} + \sum_{k=1}^{m-1}(-1)^k \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m+n)(m+n-1)\ldots(m+n-k+1)}\,a^{2k}x^{2m-2k-1}\Bigg] + (-1)^m \frac{(2m-1)!!\,a^{2m}}{2^m(m+n+1)(m+n)\ldots(n+2)}\int (x^2+a^2)^{n+1/2}\,dx.$$

**3.** $$\int x^{2m+1}(x^2+a^2)^{n+1/2}\,dx = \sum_{k=0}^{m} C_m^k \frac{(-a^2)^{m-k}(x^2+a^2)^{n+k+3/2}}{2n+2k+3}.$$

**4.** $$\int x(x^2+a^2)^{n+1/2}\,dx = \frac{1}{2n+3}\,(x^2+a^2)^{n+3/2}.$$

**5.** $$\int (x^2+a^2)^{n+1/2}\,dx = \frac{x(x^2+a^2)^{n+1/2}}{2(n+1)} + \frac{(2n+1)a^2}{2(n+1)}\int (x^2+a^2)^{n-1/2}\,dx.$$

6. $$= \frac{x(x^2+a^2)^{1/2}}{2(n+1)} \times$$
$$\times \left[(x^2+a^2)^n + \sum_{k=1}^{n} \frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2^k n(n-1)\ldots(n-k+1)}\, a^{2k}(x^2+a^2)^{n-k}\right] +$$
$$+ \frac{(2n+1)!!\, a^{2n+2}}{2^{n+1}(n+1)!} \ln\left[x + (x^2+a^2)^{1/2}\right].$$

7. $$\int x^{2m}(x^2+a^2)^{1/2}\,dx = \frac{(x^2+a^2)^{3/2}}{2(m+1)} \times$$
$$\times \left[x^{2m-1} + \sum_{k=1}^{m-1} (-1)^k \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k m(m-1)\ldots(m-k+1)}\, a^{2k} x^{2m-2k-1}\right] +$$
$$+ \frac{(2m-1)!!\,(-a^2)^m}{2^m(m+1)!} \int (x^2+a^2)^{1/2}\,dx.$$

8. $$\int (x^2+a^2)^{1/2}\,dx = \frac{x}{2}(x^2+a^2)^{1/2} + \frac{a^2}{2}\ln\left[x+(x^2+a^2)^{1/2}\right].$$

9. $$\int x(x^2+a^2)^{1/2}\,dx = \frac{1}{3}(x^2+a^2)^{3/2}.$$

10. $$\int x^2(x^2+a^2)^{1/2}\,dx =$$
$$= \frac{x}{4}(x^2+a^2)^{3/2} - \frac{a^2 x}{8}(x^2+a^2)^{1/2} - \frac{a^4}{8}\ln\left[x+(x^2+a^2)^{1/2}\right].$$

11. $$\int x^3(x^2+a^2)^{1/2}\,dx = \frac{1}{5}(x^2+a^2)^{5/2} - \frac{a^2}{3}(x^2+a^2)^{3/2}.$$

12. $$\int (x^2+a^2)^{3/2}\,dx =$$
$$= \frac{x}{4}(x^2+a^2)^{3/2} + \frac{3a^2 x}{8}(x^2+a^2)^{1/2} + \frac{3a^4}{8}\ln\left[x+(x^2+a^2)^{1/2}\right].$$

13. $$\int x(x^2+a^2)^{3/2}\,dx = \frac{1}{5}(x^2+a^2)^{5/2}.$$

14. $$\int x^2(x^2+a^2)^{3/2}\,dx = \frac{x}{6}(x^2+a^2)^{5/2} - \frac{a^2 x}{24}(x^2+a^2)^{3/2} -$$
$$- \frac{a^4 x}{16}(x^2+a^2)^{1/2} - \frac{a^6}{16}\ln\left[x+(x^2+a^2)^{1/2}\right].$$

15. $$\int x^3(x^2+a^2)^{3/2}\,dx = \frac{1}{7}(x^2+a^2)^{7/2} \mp \frac{a^2}{5}(x^2+a^2)^{5/2}.$$

**2.41.** Интегралы вида $\int \frac{(x^2+a^2)^{n+1/2}}{x^m}\,dx$.

1. $$\int \frac{(x^2+a^2)^{n+1/2}}{x^m}\,dx = -\frac{(x^2+a^2)^{n+3/2}}{(m-1)a^2 x^{m-1}} +$$
$$+ \frac{2n-m+4}{(m-1)a^2}\int \frac{(x^2+a^2)^{n+1/2}}{x^{m-2}}\,dx.$$

2. $$= -\frac{(x^2+a^2)^{n+1/2}}{(m-1)x^{m-1}} + \frac{2n+1}{m-1}\int \frac{(x^2+a^2)^{n-1/2}}{x^{m-2}}\,dx.$$

3. $$\int \frac{(x^2+a^2)^{n+1/2}}{x^{2m}}\,dx = -\frac{(x^2+a^2)^{n+3/2}}{(2m-1)a^2}\left[\frac{1}{x^{2m-1}} + \sum_{k=1}^{m-1} \frac{(-2)^k(m-n-2)(m-n-3)\ldots(m-n-k-1)}{(2m-3)(2m-5)\ldots(2m-2k-1)a^{2k}x^{2m-2k-1}}\right] + \frac{(-2)^m(m-n-2)(m-n-3)\ldots(-n)(-n-1)}{(2m-1)!!\,a^{2m}}\int (x^2+a^2)^{n+1/2}\,dx.$$

4. $$= \frac{(x^2+a^2)^{n+3/2}}{(-a^2)^{m-n-1}x^{2n+3}} \sum_{k=0}^{m-n-2} \frac{(-1)^k C_{m-n-2}^k}{2n+2k+3}\left(\frac{x^2+a^2}{x^2}\right)^k$$

$[m \geqslant n+2]$.

5. $$\int \frac{(x^2+a^2)^{n+1/2}}{x^{2n}}\,dx = -\sum_{k=0}^{n-1} \frac{(2n+1)(x^2+a^2)^{n-k+1/2}}{[(2n-2k)^2-1]x^{2n-2k-1}} + (2n+1)\int (x^2+a^2)^{1/2}\,dx.$$

6. $$\int \frac{(x^2+a^2)^{n+1/2}}{x^{2m+1}}\,dx = -\frac{(x^2+a^2)^{n+3/2}}{2n-2m+1} \times \sum_{k=1}^{m} \frac{(2n-2m+1)(2n-2m+3)\ldots(2n-2m+2k-1)}{2^k m(m-1)\ldots(m-k+1)a^{2k}x^{2m-2k+2}} + \frac{(2n-2m+3)(2n-2m+5)\ldots(2n+1)}{m!\,(2a^2)^m}\int \frac{(x^2+a^2)^{n+1/2}}{x}\,dx.$$

7. $$\int \frac{(x^2+a^2)^{n+1/2}}{x}\,dx = \sum_{k=0}^{n} \frac{a^{2k}(x^2+a^2)^{n-k+1/2}}{2n-2k+1} + \frac{a^{2n+1}}{2}\ln\frac{(x^2+a^2)^{1/2}-a}{(x^2+a^2)^{1/2}+a}.$$

8. $$\int \frac{(x^2+a^2)^{1/2}}{x}\,dx = (x^2+a^2)^{1/2} - a\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

9. $$\int \frac{(x^2+a^2)^{1/2}}{x^2}\,dx = -\frac{(x^2+a^2)^{1/2}}{x} + \ln[x+(x^2+a^2)^{1/2}].$$

10. $$\int \frac{(x^2+a^2)^{1/2}}{x^3}\,dx = -\frac{(x^2+a^2)^{1/2}}{2x^2} + \frac{1}{2a}\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

11. $$\int \frac{(x^2+a^2)^{3/2}}{x}\,dx = \frac{1}{3}(x^2+a^2)^{3/2} + a^2(x^2+a^2)^{1/2} - a^3\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

12. $$\int \frac{(x^2+a^2)^{3/2}}{x^2}\,dx = -\frac{(x^2+a^2)^{3/2}}{x} + \frac{3}{2}x(x^2+a^2)^{1/2} + \frac{3a^2}{2}\ln[x+(x^2+a^2)^{1/2}].$$

13. $$\int \frac{(x^2+a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2+a^2)^{3/2}}{2x^2} + \frac{3}{2}(x^2+a^2)^{1/2} - \frac{3a}{2}\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

**14.** $$\int \frac{(x^2+a^2)^{n+1/2}}{x^{2n+2}}\,dx = -\sum_{k=0}^{n} \frac{1}{2n-2k+1}\left(\frac{x^2+a^2}{x^2}\right)^{n-k+1/2} + \ln\left[x+(x^2+a^2)^{1/2}\right].$$

**15.** $$\int \frac{(x^2+a^2)^{n+1/2}}{x^{2n+4}}\,dx = -\frac{(x^2+a^2)^{n+3/2}}{(2n+3)a^2x^{2n+3}}.$$

## 2.42. Интегралы вида $\int \frac{x^m\,dx}{(x^2+a^2)^{n+1/2}}$.

**1.** $$\int \frac{x^m\,dx}{(x^2+a^2)^{n+1/2}} = -\frac{x^{m-1}}{(2n-1)(x^2+a^2)^{n-1/2}} + \frac{m-1}{2n-1}\int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n-1/2}}.$$

**2.** $$= \frac{x^{m+1}}{(2n-1)a^2(x^2+a^2)^{n-1/2}} + \frac{2n-m-2}{(2n-1)a^2}\int \frac{x^m\,dx}{(x^2+a^2)^{n-1/2}}.$$

**3.** $$= \frac{x^{m-1}}{(m-2n)(x^2+a^2)^{n-1/2}} - \frac{(m-1)a^2}{m-2n}\int \frac{x^{m-2}\,dx}{(x^2+a^2)^{n+1/2}}.$$

**4.** $$\int \frac{x^{2m}\,dx}{(x^2+a^2)^{n+1/2}} = \frac{x^{2m+1}(x^2+a^2)^{1/2}}{(2n-1)a^2}\left[\frac{1}{(x^2+a^2)^n} + \sum_{k=1}^{n-1} \frac{2^k(n-m-1)(n-m-2)\ldots(n-m-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\,\frac{a^{-2k}}{(x^2+a^2)^{n-k}}\right] + \frac{2^n(n-m-1)(n-m-2)\ldots(-m+1)(-m)}{(2n-1)!!\,a^{2n}}\int \frac{x^{2m}\,dx}{(x^2+a^2)^{1/2}}.$$

**5.** $$= a^{2m-2n}\sum_{k=0}^{n-m-1} \frac{(-1)^k C_{n-m-1}^k}{2m+2k+1}\left(\frac{x^2}{x^2+a^2}\right)^{m+k+1/2} \qquad [n \geqslant m+1].$$

**6.** $$\int \frac{x^{2m}\,dx}{(x^2+a^2)^{1/2}} = \frac{(x^2+a^2)^{1/2}}{2m} \times \left[x^{2m-1} - \sum_{k=1}^{m-1} \frac{(-1)^k(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m-1)(m-2)\ldots(m-k)}\,a^{2k}x^{2m-2k-1}\right] + (-a^2)^m\frac{(2m-1)!!}{2^m m!}\ln\left[x+(x^2+a^2)^{1/2}\right].$$

**7.** $$\int \frac{x^{2n-2}\,dx}{(x^2+a^2)^{n+1/2}} = \frac{x^{2n-1}}{(2n-1)(x^2+a^2)^{n-1/2}}.$$

**8.** $$\int \frac{x^{2m+1}\,dx}{(x^2+a^2)^{n+1/2}} = \sum_{k=0}^{m} \frac{(-1)^{m+k+1}C_m^k a^{2m-2k}}{(2n-2k-1)(x^2+a^2)^{n-k-1/2}}.$$

**9.** $$\int \frac{dx}{(x^2+a^2)^{n+1/2}} = \frac{x}{(2n-1)a^2(x^2+a^2)^{n-1/2}} + \frac{2n-2}{(2n-1)a^2}\int \frac{dx}{(x^2+a^2)^{n-1/2}}.$$

**10.** $$= \frac{x(x^2+a^2)^{1/2}}{(2n-1)a^2}\left[\frac{1}{(x^2+a^2)^n} + \sum_{k=1}^{n-1} \frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\cdot\frac{1}{a^{2k}(x^2+a^2)^{n-k}}\right].$$

**11.** $$= \frac{1}{a^{2n}} \sum_{k=0}^{n-1} \frac{(-1)^k}{2k+1} C_{n-1}^k \left( \frac{x^2}{x^2+a^2} \right)^{k+1/2}.$$

**12.** $$\int \frac{x\,dx}{(x^2+a^2)^{n+1/2}} = -\frac{1}{(2n-1)(x^2+a^2)^{n-1/2}}.$$

**13.** $$\int \frac{dx}{(x^2+a^2)^{1/2}} = \ln\left[x + (x^2+a^2)^{1/2}\right].$$

**14.** $$\int \frac{x\,dx}{(x^2+a^2)^{1/2}} = (x^2+a^2)^{1/2}.$$

**15.** $$\int \frac{x^2\,dx}{(x^2+a^2)^{1/2}} = \frac{x(x^2+a^2)^{1/2}}{2} - \frac{a^2}{2} \ln\left[x + (x^2+a^2)^{1/2}\right].$$

**16.** $$\int \frac{x^3\,dx}{(x^2+a^2)^{1/2}} = \frac{1}{3}(x^2+a^2)^{3/2} - a^2(x^2+a^2)^{1/2}.$$

**17.** $$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2(x^2+a^2)^{1/2}}.$$

**18.** $$\int \frac{x\,dx}{(x^2+a^2)^{3/2}} = -\frac{1}{(x^2+a^2)^{1/2}}.$$

**19.** $$\int \frac{x^2\,dx}{(x^2+a^2)^{3/2}} = -\frac{x}{(x^2+a^2)^{1/2}} + \ln\left[x + (x^2+a^2)^{1/2}\right].$$

**20.** $$\int \frac{x^3\,dx}{(x^2+a^2)^{3/2}} = \frac{x^2+2a^2}{(x^2+a^2)^{1/2}}.$$

### 2.43. Интегралы вида $\int \frac{dx}{x^m(x^2+a^2)^{n+1/2}}$.

**1.** $$\int \frac{dx}{x^m(x^2+a^2)^{n+1/2}} =$$
$$= \frac{1}{(2n-1)a^2x^{m-1}(x^2+a^2)^{n-1/2}} + \frac{m+2n-2}{(2n-1)a^2} \int \frac{dx}{x^m(x^2+a^2)^{n-1/2}}.$$

**2.** $$= -\frac{1}{(2n-1)x^{m+1}(x^2+a^2)^{n-1/2}} - \frac{m+1}{2n-1} \int \frac{dx}{x^{m+2}(x^2+a^2)^{n-1/2}}.$$

**3.** $$= -\frac{1}{(m-1)x^{m-1}(x^2+a^2)^{n+1/2}} - \frac{2n-1}{m-1} \int \frac{dx}{x^{m-2}(x^2+a^2)^{n+3/2}}.$$

**4.** $$\int \frac{dx}{x^{2m}(x^2+a^2)^{n+1/2}} = \frac{1}{a^{2m+2n}} \times$$
$$\times \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2k-2m+1} C_{m+n-1}^k \left( \frac{x^2}{x^2+a^2} \right)^{k-m+1/2}.$$

**5.** $$\int \frac{dx}{x(x^2+a^2)^{n+1/2}} = \frac{1}{(2n-1)a^2(x^2+a^2)^{n-1/2}} + \frac{1}{a^2} \int \frac{dx}{x(x^2+a^2)^{n-1/2}}.$$

**6.** $$= \sum_{k=0}^{n-1} \frac{1}{(2n-2k-1)a^{2k+2}(x^2+a^2)^{n-k-1/2}} -$$
$$- \frac{1}{a^{2n+1}} \ln \frac{a + (x^2+a^2)^{1/2}}{|x|}.$$

**7.** $$\int \frac{dx}{x(x^2+a^2)^{1/2}} = -\frac{1}{a}\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

**8.** $$\int \frac{dx}{x^2(x^2+a^2)^{1/2}} = -\frac{(x^2+a^2)^{1/2}}{a^2x}.$$

**9.** $$\int \frac{dx}{x^3(x^2+a^2)^{1/2}} = -\frac{(x^2+a^2)^{1/2}}{2a^2x^2} + \frac{1}{2a^3}\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

**10.** $$\int \frac{dx}{x(x^2+a^2)^{3/2}} = \frac{1}{a^2(x^2+a^2)^{1/2}} - \frac{1}{a^3}\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

**11.** $$\int \frac{dx}{x^2(x^2+a^2)^{3/2}} = -\frac{2x^2+a^2}{a^4x(x^2+a^2)^{1/2}}.$$

**12.** $$\int \frac{dx}{x^3(x^2+a^2)^{3/2}} = \\ = -\frac{1}{2a^2x^2(x^2+a^2)^{1/2}} - \frac{3}{2a^4(x^2+a^2)^{1/2}} + \frac{3}{2a^5}\ln\frac{a+(x^2+a^2)^{1/2}}{|x|}.$$

### 2.44. Интегралы вида $\int \frac{dx}{(x+b)^n(x^2+a^2)^{1/2}}$.

**1.** $$\int \frac{dx}{(x+b)^n(x^2+a^2)^{1/2}} = -\frac{1}{(n-1)(a^2+b^2)}\left[\frac{(x^2+a^2)^{1/2}}{(x+b)^{n-1}} - \right. \\ \left. -(2n-3)b\int\frac{dx}{(x+b)^{n-1}(x^2+a^2)^{1/2}} + (n-2)\int\frac{dx}{(x+b)^{n-2}(x^2+a^2)^{1/2}}\right].$$

**2.** $$\int \frac{dx}{(x+b)(x^2+a^2)^{1/2}} = \\ = \frac{1}{(a^2+b^2)^{1/2}}\ln\left|\frac{a^2-bx-(a^2+b^2)^{1/2}(x^2+a^2)^{1/2}}{x+b}\right|.$$

### 2.45. Интегралы вида $\int \frac{dx}{(x^2\pm b^2)^m(x^2+a^2)^{n+1/2}}$.

**1.** $$\int \frac{dx}{(x^2+b^2)^m(x^2+a^2)^{n+1/2}} = \\ = \frac{x}{2(m-1)(a^2-b^2)b^2(x^2+b^2)^{m-1}(x^2+a^2)^{n-1/2}} + \\ + \frac{(2m-3)a^2-(4m+2n-6)b^2}{2(m-1)(a^2-b^2)b^2}\int\frac{dx}{(x^2+b^2)^{m-1}(x^2+a^2)^{n+1/2}} + \\ + \frac{m+n-2}{(m-1)(a^2-b^2)b^2}\int\frac{dx}{(x^2+b^2)^{m-2}(x^2+a^2)^{n+1/2}}.$$

**2.** $$\int \frac{dx}{(x^2+b^2)(x^2+a^2)^{n+1/2}} = -\frac{x}{(2n-1)(a^2-b^2)a^2(x^2+a^2)^{n-1/2}} + \\ + \frac{(4n-3)a^2-(2n-2)b^2}{(2n-1)(a^2-b^2)a^2}\int\frac{dx}{(x^2+b^2)(x^2+a^2)^{n-1/2}} - \\ - \frac{2n-2}{(2n-1)(a^2-b^2)a^2}\int\frac{dx}{(x^2+b^2)(x^2+a^2)^{n-3/2}}.$$

**3.** $$\int \frac{dx}{(x^2+b^2)(x^2+a^2)^{1/2}} = \frac{1}{b(b^2-a^2)^{1/2}} \ln \frac{\left|x(b^2-a^2)^{1/2}+b(x^2+a^2)^{1/2}\right|}{(x^2+b^2)^{1/2}} \qquad [b^2>a^2].$$

**4.** $$= \frac{1}{b(a^2-b^2)^{1/2}} \operatorname{arctg} \frac{x(a^2-b^2)^{1/2}}{b(x^2+a^2)^{1/2}} \qquad [a^2>b^2].$$

**5.** $$\int \frac{dx}{(x^2-b^2)(x^2+a^2)^{1/2}} = \frac{1}{2b(a^2+b^2)^{1/2}} \ln \left|\frac{x(a^2+b^2)^{1/2}-b(x^2+a^2)^{1/2}}{x(a^2+b^2)^{1/2}+b(x^2+a^2)^{1/2}}\right|.$$

## 2.46. Интегралы вида $\int x^m (x^2-a^2)^{n+1/2}\,dx$.

**1.** $$\int x^m (x^2-a^2)^{n+1/2}\,dx = \frac{x^{m-1}(x^2-a^2)^{n+3/2}}{m+2n+2} + \frac{(m-1)a^2}{m+2n+2}\int x^{m-2}(x^2-a^2)^{n+1/2}\,dx.$$

**2.** $$\int x^{2m}(x^2-a^2)^{n+1/2}\,dx = \frac{(x^2-a^2)^{n+3/2}}{2m+2n+2}\left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m+n)(m+n-1)\ldots(m+n-k+1)}\, a^{2k} x^{2m-2k-1}\right] + \frac{(2m-1)!!\,a^{2m}}{2^m(m+n+1)(m+n)\ldots(n+2)}\int (x^2-a^2)^{n+1/2}\,dx.$$

**3.** $$\int x^{2m+1}(x^2-a^2)^{n+1/2}\,dx = \sum_{k=0}^{m} C_m^k \frac{a^{2(m-k)}(x^2-a^2)^{n+k+3/2}}{2n+2k+3}.$$

**4.** $$\int x(x^2-a^2)^{n+1/2}\,dx = \frac{1}{2n+3}(x^2-a^2)^{n+3/2}.$$

**5.** $$\int (x^2-a^2)^{n+1/2}\,dx = \frac{x(x^2-a^2)^{n+1/2}}{2(n+1)} - \frac{(2n+1)a^2}{2(n+1)}\int (x^2-a^2)^{n-1/2}\,dx.$$

**6.** $$= \frac{x(x^2-a^2)^{1/2}}{2(n+1)} \times \left[(x^2-a^2)^n + \sum_{k=1}^{n} (-1)^k \frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2^k n(n-1)\ldots(n-k+1)}\, a^{2k}(x^2-a^2)^{n-k}\right] + \frac{(2n+1)!!\,(-a^2)^{n+1}}{2^{n+1}(n+1)!} \ln\left|x+(x^2-a^2)^{1/2}\right|.$$

**7.** $$\int x^{2m}(x^2-a^2)^{1/2}\,dx = \frac{(x^2-a^2)^{3/2}}{2(m+1)} \times \left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k m(m-1)\ldots(m-k+1)}\, a^{2k} x^{2m-2k-1}\right] + \frac{(2m-1)!!\,a^{2m}}{2^m(m+1)!}\int (x^2-a^2)^{1/2}\,dx.$$

**8.** $$\int (x^2-a^2)^{1/2}\,dx = \frac{x}{2}(x^2-a^2)^{1/2} - \frac{a^2}{2}\ln|x+(x^2-a^2)^{1/2}|.$$

**9.** $$\int x(x^2-a^2)^{1/2}\,dx = \frac{1}{3}(x^2-a^2)^{3/2}.$$

**10.** $$\int x^2(x^2-a^2)^{1/2}\,dx = \frac{x}{4}(x^2-a^2)^{3/2} + \frac{a^2x}{8}(x^2-a^2)^{1/2} - \frac{a^4}{8}\ln|x+(x^2-a^2)^{1/2}|.$$

**11.** $$\int x^3(x^2-a^2)^{1/2}\,dx = \frac{1}{5}(x^2-a^2)^{5/2} + \frac{a^2}{3}(x^2-a^2)^{3/2}.$$

**12.** $$\int (x^2-a^2)^{3/2}\,dx = \frac{x}{4}(x^2-a^2)^{3/2} - \frac{3a^2x}{8}(x^2-a^2)^{1/2} + \frac{3}{8}a^4\ln|x+(x^2-a^2)^{1/2}|.$$

**13.** $$\int x(x^2-a^2)^{3/2}\,dx = \frac{1}{5}(x^2-a^2)^{5/2}.$$

**14.** $$\int x^2(x^2-a^2)^{3/2}\,dx = \frac{x}{6}(x^2-a^2)^{5/2} + \frac{a^2x}{24}(x^2-a^2)^{3/2} - \frac{a^4x}{16}(x^2-a^2)^{1/2} + \frac{a^6}{16}\ln|x+(x^2-a^2)^{1/2}|.$$

**15.** $$\int x^3(x^2-a^2)^{3/2}\,dx = \frac{1}{7}(x^2-a^2)^{7/2} + \frac{a^2}{5}(x^2-a^2)^{5/2}.$$

### **2.47.** Интегралы вида $\int \frac{(x^2-a^2)^{n+1/2}}{x^m}\,dx$.

**1.** $$\int \frac{(x^2-a^2)^{n+1/2}}{x^m}\,dx = \frac{(x^2-a^2)^{n+3/2}}{(m-1)a^2x^{m-1}} + \frac{m-2n-4}{(m-1)a^2}\int \frac{(x^2-a^2)^{n+1/2}}{x^{m-2}}\,dx.$$

**2.** $$= -\frac{(x^2-a^2)^{n+1/2}}{(m-1)x^{m-1}} + \frac{2n-1}{m-1}\int \frac{(x^2-a^2)^{n-1/2}}{x^{m-2}}\,dx.$$

**3.** $$\int \frac{(x^2-a^2)^{n+1/2}}{x^{2m}}\,dx = \frac{(x^2-a^2)^{n+3/2}}{(2m-1)a^2}\left[\frac{1}{x^{2m-1}} + \sum_{k=1}^{m-1}\frac{2^k(m-n-2)(m-n-3)\ldots(m-n-k-1)}{(2m-3)(2m-5)\ldots(2m-2k-1)a^{2k}}\cdot\frac{1}{x^{2m-2k-1}}\right] + \frac{2^m(m-n-2)(m-n-3)\ldots(-n)(-n-1)}{(2m-1)!!\,a^{2m}}\int (x^2-a^2)^{n+1/2}\,dx.$$

**4.** $$= \frac{(x^2-a^2)^{n+3/2}}{a^{2(m-n-1)}x^{2n+3}}\sum_{k=0}^{m-n-2}\frac{(-1)^kC_{m-n-2}^k}{2n+2k+3}\left(\frac{x^2-a^2}{x^2}\right)^k$$

$[m \geqslant n+2]$.

**5.** $$\int \frac{(x^2-a^2)^{n+1/2}}{x^{2n}}\,dx = -\sum_{k=0}^{n-1}\frac{(2n+1)(x^2-a^2)^{n-k+1/2}}{[(2n-2k)^2-1]x^{2n-2k-1}} + (2n+1)\int (x^2-a^2)^{1/2}\,dx.$$

6. $$\int \frac{(x^2-a^2)^{n+1/2}}{x^{2m+1}}\,dx = -\frac{(x^2-a^2)^{n+3/2}}{2n-2m+1}\times$$
$$\times\sum_{k=1}^{m}(-1)^k\frac{(2n-2m+1)(2n-2m+3)\ldots(2n-2m+2k-1)}{2^k m(m-1)\ldots(m-k+1)a^{2k}x^{2m-2k+2}}+$$
$$+(-1)^m\frac{(2n-2m+3)(2n-2m+5)\ldots(2n+1)}{m!\,(2a^2)^m}\int\frac{(x^2-a^2)^{n+1/2}}{x}\,dx.$$

7. $$\int \frac{(x^2-a^2)^{n+1/2}}{x}\,dx =$$
$$=\sum_{k=0}^{n}\frac{(-a^2)^k(x^2-a^2)^{n-k+1/2}}{2n-2k+1}+(-1)^n a^{2n+1}\arcsin\left|\frac{a}{x}\right|.$$

8. $$\int \frac{(x^2-a^2)^{1/2}}{x}\,dx = (x^2-a^2)^{1/2} - a\arccos\left|\frac{a}{x}\right|.$$

9. $$\int \frac{(x^2-a^2)^{1/2}}{x^2}\,dx = -\frac{(x^2-a^2)^{1/2}}{x} + \ln\left|x+(x^2-a^2)^{1/2}\right|.$$

10. $$\int \frac{(x^2-a^2)^{1/2}}{x^3}\,dx = -\frac{(x^2-a^2)^{1/2}}{2x^2} + \frac{1}{2a}\arccos\left|\frac{a}{x}\right|.$$

11. $$\int \frac{(x^2-a^2)^{3/2}}{x}\,dx = \frac{1}{3}(x^2-a^2)^{3/2} - a^2(x^2-a^2)^{1/2} + a^3\arccos\left|\frac{a}{x}\right|.$$

12. $$\int \frac{(x^2-a^2)^{3/2}}{x^2}\,dx =$$
$$= -\frac{(x^2-a^2)^{3/2}}{x} + \frac{3x}{2}(x^2-a^2)^{1/2} - \frac{3a^2}{2}\ln\left|x+(x^2-a^2)^{1/2}\right|.$$

13. $$\int \frac{(x^2-a^2)^{3/2}}{x^3}\,dx = -\frac{(x^2-a^2)^{3/2}}{2x^2} + \frac{3}{2}(x^2-a^2)^{1/2} - \frac{3a}{2}\arccos\left|\frac{a}{x}\right|.$$

14. $$\int \frac{(x^2-a^2)^{n+1/2}}{x^{2n+2}}\,dx =$$
$$= -\sum_{k=0}^{n}\frac{1}{2n-2k+1}\left(\frac{x^2-a^2}{x^2}\right)^{n-k+1/2} + \ln\left|x+(x^2-a^2)^{1/2}\right|.$$

15. $$\int \frac{(x^2-a^2)^{n+1/2}}{x^{2n+4}}\,dx = \frac{(x^2-a^2)^{n+3/2}}{(2n+3)a^2x^{2n+3}}.$$

**2.48. Интегралы вида** $\int\frac{x^m\,dx}{(x^2-a^2)^{n+1/2}}$.

1. $$\int \frac{x^m\,dx}{(x^2-a^2)^{n+1/2}} = -\frac{x^{m-1}}{(2n-1)(x^2-a^2)^{n-1/2}} + \frac{m-1}{2n-1}\int\frac{x^{m-2}\,dx}{(x^2-a^2)^{n-1/2}}.$$

2. $$= -\frac{x^{m+1}}{(2n-1)a^2(x^2-a^2)^{n-1/2}} + \frac{m-2n+2}{(2n-1)a^2}\int\frac{x^m\,dx}{(x^2-a^2)^{n-1/2}}.$$

3. $$= \frac{x^{m-1}}{(m-2n)(x^2-a^2)^{n-1/2}} + \frac{(m-1)a^2}{m-2n}\int\frac{x^{m-2}\,dx}{(x^2-a^2)^{n+1/2}}.$$

**4.** $$\int \frac{x^{2m}\,dx}{(x^2-a^2)^{n+1/2}} = \frac{x^{2m+1}(x^2-a^2)^{1/2}}{(2n-1)a^2}\left[\frac{1}{(x^2-a^2)^n} + \sum_{k=1}^{n-1}\frac{(-2)^k(n-m-1)(n-m-2)\ldots(n-m-k)a^{-2k}}{(2n-3)(2n-5)\ldots(2n-2k-1)(x^2-a^2)^{n-k}}\right] + \frac{(-2)^n(n-m-1)(n-m-2)\ldots(-m+1)(-m)}{(2n-1)!!\,a^{2n}}\int\frac{x^{2m}\,dx}{(x^2-a^2)^{1/2}}.$$

**5.** $$= (-a^2)^{m-n}\sum_{k=0}^{n-m-1}\frac{(-1)^k C_{n-m-1}^k}{2m+2k+1}\left(\frac{x^2}{x^2-a^2}\right)^{m+k+1/2}$$

$[n \geqslant m+1]$.

**6.** $$\int \frac{x^{2m}\,dx}{(x^2-a^2)^{1/2}} = \frac{(x^2-a^2)^{1/2}}{2m}\times\left[x^{2m-1} + \sum_{k=1}^{m-1}\frac{(2m-1)(2m-3)\ldots(2m-2k-1)}{2^k(m-1)(m-2)\ldots(m-k)}a^{2k}x^{2m-2k+1}\right] + \frac{(2m-1)!!\,a^{2m}}{2^m m!}\ln\left|x+(x^2-a^2)^{1/2}\right|.$$

**7.** $$\int \frac{x^{2n-2}\,dx}{(x^2-a^2)^{n+1/2}} = -\frac{x^{2n-1}}{(2n-1)(x^2-a^2)^{n-1/2}}.$$

**8.** $$\int \frac{x^{2m+1}\,dx}{(x^2-a^2)^{n+1/2}} = -\sum_{k=0}^{m}\frac{C_m^k a^{2m-2k}}{(2n-2k-1)(x^2-a^2)^{n-k-1/2}}.$$

**9.** $$\int \frac{dx}{(x^2-a^2)^{n+1/2}} = -\frac{x}{(2n-1)a^2(x^2-a^2)^{n-1/2}} - \frac{2n-2}{(2n-1)a^2}\int\frac{dx}{(x^2-a^2)^{n-1/2}}.$$

**10.** $$= -\frac{x(x^2-a^2)^{1/2}}{(2n-1)a^2}\left[\frac{1}{(x^2-a^2)^n} + \sum_{k=1}^{n-1}\frac{(-2)^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\cdot\frac{1}{a^{2k}(x^2-a^2)^{n-k}}\right].$$

**11.** $$= \frac{(-1)^n}{a^{2n}}\sum_{k=0}^{n-1}\frac{(-1)^k}{2k+1}C_{n-1}^k\left(\frac{x^2}{x^2-a^2}\right)^{k+1/2}.$$

**12.** $$\int \frac{x\,dx}{(x^2-a^2)^{n+1/2}} = -\frac{1}{(2n-1)(x^2-a^2)^{n-1/2}}.$$

**13.** $$\int \frac{dx}{(x^2-a^2)^{1/2}} = \ln\left|x+(x^2-a^2)^{1/2}\right|.$$

**14.** $$\int \frac{x\,dx}{(x^2-a^2)^{1/2}} = (x^2-a^2)^{1/2}.$$

**15.** $$\int \frac{x^2\,dx}{(x^2-a^2)^{1/2}} = \frac{x(x^2-a^2)^{1/2}}{2} + \frac{a^2}{2}\ln\left|x+(x^2-a^2)^{1/2}\right|.$$

**16.** $$\int \frac{x^3\,dx}{(x^2-a^2)^{1/2}} = \frac{1}{3}(x^2-a^2)^{3/2} + a^2(x^2-a^2)^{1/2}.$$

**17.** $\displaystyle\int \frac{dx}{(x^2-a^2)^{3/2}} = -\frac{x}{a^2(x^2-a^2)^{1/2}}.$

**18.** $\displaystyle\int \frac{x\,dx}{(x^2-a^2)^{3/2}} = -\frac{1}{(x^2-a^2)^{1/2}}.$

**19.** $\displaystyle\int \frac{x^2\,dx}{(x^2-a^2)^{3/2}} = -\frac{x}{(x^2-a^2)^{1/2}} + \ln\left|x+(x^2-a^2)^{1/2}\right|.$

**20.** $\displaystyle\int \frac{x^3\,dx}{(x^2-a^2)^{3/2}} = \frac{x^2-2a^2}{(x^2-a^2)^{1/2}}.$

**2.49. Интегралы вида** $\displaystyle\int \frac{dx}{x^m(x^2-a^2)^{n+1/2}}$.

**1.** $\displaystyle\int \frac{dx}{x^m(x^2-a^2)^{n+1/2}} =$

$$= -\frac{1}{(2n-1)a^2x^{m-1}(x^2-a^2)^{n-1/2}} - \frac{m+2n-2}{(2n-1)a^2}\int \frac{dx}{x^m(x^2-a^2)^{n-1/2}}.$$

**2.** $$= -\frac{1}{(2n-1)x^{m+1}(x^2-a^2)^{n-1/2}} - \frac{m+1}{2n-1}\int \frac{dx}{x^{m+2}(x^2-a^2)^{n-1/2}}.$$

**3.** $$= -\frac{1}{(m-1)x^{m-1}(x^2-a^2)^{n+1/2}} - \frac{2n-1}{m-1}\int \frac{dx}{x^{m-2}(x^2-a^2)^{n+3/2}}.$$

**4.** $\displaystyle\int \frac{dx}{x^{2m}(x^2-a^2)^{n+1/2}} = \frac{1}{a^{2m+2n}} \times$

$$\times \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2k-2m+1} C_{m+n-1}^k \left(\frac{x^2}{x^2-a^2}\right)^{k-m+1/2}.$$

**5.** $\displaystyle\int \frac{dx}{x(x^2-a^2)^{n+1/2}} = -\frac{1}{(2n-1)a^2(x^2-a^2)^{n-1/2}} - \frac{1}{a^2}\int \frac{dx}{x(x^2-a^2)^{n-1/2}}.$

**6.** $$= \sum_{k=0}^{n-1} \frac{(-1)^{k+1}}{(2n-2k-1)a^{2k+2}(x^2-a^2)^{n-k-1/2}} - \frac{(-1)^{n+1}}{a^{2n+1}}\arccos\left|\frac{a}{x}\right|.$$

**7.** $\displaystyle\int \frac{dx}{x(x^2-a^2)^{1/2}} = \frac{1}{a}\arccos\left|\frac{a}{x}\right|.$

**8.** $\displaystyle\int \frac{dx}{x^2(x^2-a^2)^{1/2}} = \frac{(x^2-a^2)^{1/2}}{a^2x}.$

**9.** $\displaystyle\int \frac{dx}{x^3(x^2-a^2)^{1/2}} = \frac{(x^2-a^2)^{1/2}}{2a^2x^2} + \frac{1}{2a^3}\arccos\left|\frac{a}{x}\right|.$

**10.** $\displaystyle\int \frac{dx}{x(x^2-a^2)^{3/2}} = -\frac{1}{a^2(x^2-a^2)^{1/2}} - \frac{1}{a^3}\arccos\left|\frac{a}{x}\right|.$

**11.** $\displaystyle\int \frac{dx}{x^2(x^2-a^2)^{3/2}} = -\frac{2x^2-a^2}{a^4x(x^2-a^2)^{1/2}}.$

**12.** $\displaystyle\int \frac{dx}{x^3(x^2-a^2)^{3/2}} =$

$$= \frac{1}{2a^2x^2(x^2-a^2)^{1/2}} - \frac{3}{2a^4(x^2-a^2)^{1/2}} - \frac{3}{2a^5}\arccos\left|\frac{a}{x}\right|.$$

**2.50.** Интегралы вида $\int \frac{dx}{(x+b)^n(x^2-a^2)^{1/2}}$.

**1.** $$\int \frac{dx}{(x+b)^n(x^2-a^2)^{1/2}} = -\frac{1}{(n-1)(b^2-a^2)}\left[\frac{(x^2-a^2)^{1/2}}{(x+b)^{n-1}} - {} \right.$$
$$\left. -(2n-3)b\int \frac{dx}{(x+b)^{n-1}(x^2-a^2)^{1/2}} + (n-2)\int \frac{dx}{(x+b)^{n-2}(x^2-a^2)^{1/2}}\right].$$

**2.** $$\int \frac{dx}{(x+b)(x^2-a^2)^{1/2}} =$$
$$= \frac{1}{(b^2-a^2)^{1/2}} \ln\left|\frac{a^2+bx+(b^2-a^2)^{1/2}(x^2-a^2)^{1/2}}{x+a}\right| \qquad [b^2>a^2].$$

**3.** $$= \frac{1}{(a^2-b^2)^{1/2}} \operatorname{arctg} \frac{(a^2-b^2)^{1/2}(x^2-a^2)^{1/2}}{a^2+bx} \qquad [a^2>b^2].$$

**4.** $$= \frac{1}{(a^2-b^2)^{1/2}} \arccos \frac{a^2+bx}{a(x+b)} \qquad [a^2>b^2].$$

**5.** $$\int \frac{dx}{(x+a)(x^2-a^2)^{1/2}} = \frac{1}{a}\left(\frac{x-a}{x+a}\right)^{1/2}.$$

**6.** $$\int \frac{dx}{(x-a)(x^2-a^2)^{1/2}} = -\frac{1}{a}\left(\frac{x+a}{x-a}\right)^{1/2}.$$

**2.51.** Интегралы вида $\int \frac{dx}{(x^2\pm b^2)(x^2-a^2)^{1/2}}$.

**1.** $$\int \frac{dx}{(x^2+b^2)(x^2-a^2)^{1/2}} = \frac{1}{b(a^2+b^2)^{1/2}} \ln \frac{\left|x(a^2+b^2)^{1/2}+b(x^2-a^2)^{1/2}\right|}{(x^2+b^2)^{1/2}}.$$

**2.** $$\int \frac{dx}{(x^2-b^2)(x^2-a^2)^{1/2}} = \frac{1}{b(b^2-a^2)^{1/2}} \ln \frac{\left|x(b^2-a^2)^{1/2}-b(x^2-a^2)^{1/2}\right|}{(x^2-b^2)^{1/2}} \qquad [b^2>a^2].$$

**3.** $$= \frac{1}{b(a^2-b^2)^{1/2}} \arcsin \frac{b(x^2-a^2)^{1/2}}{a(x^2-b^2)^{1/2}} \qquad [a^2>b^2].$$

**4.** $$= \frac{1}{b(a^2-b^2)^{1/2}} \operatorname{arctg} \frac{b(x^2-a^2)^{1/2}}{x(a^2-b^2)^{1/2}} \qquad [a^2>b^2].$$

**2.52.** Интегралы вида $\int x^m(a^2-x^2)^{n+1/2}\,dx$.

**1.** $$\int x^m(a^2-x^2)^{n+1/2}\,dx =$$
$$= -\frac{x^{m-1}(a^2-x^2)^{n+3/2}}{m+2n+2} + \frac{(m-1)a^2}{m+2n+2}\int x^{m-2}(a^2-x^2)^{n+1/2}\,dx.$$

**2.** $$= \frac{x^{m+1}(a^2-x^2)^{n+1/2}}{m+2n+2} + \frac{(2n+1)a^2}{m+2n+2}\int x^m(a^2-x^2)^{n-1/2}\,dx.$$

3. $\displaystyle\int x^{2m}(a^2-x^2)^{n+1/2}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{2n+2m+2}\times$

$$\times\left[x^{2m-1}+\sum_{k=1}^{m-1}\frac{(2m-1)(2m-3)\dots(2m-2k+1)}{2^k(m+n)(m+n-1)\dots(m+n-k+1)}\,a^{2k}x^{2m-2k-1}\right]+$$

$$+\frac{(2m-1)!!\,a^{2m}}{2^m(m+n+1)(m+n)\dots(n+2)}\int(a^2-x^2)^{n+1/2}\,dx.$$

4. $\displaystyle\int x^{2m+1}(a^2-x^2)^{n+1/2}\,dx = \sum_{k=0}^{m}\frac{(-1)^{k+1}C_m^k a^{2m-2k}}{2n+2k+3}(a^2-x^2)^{n+k+3/2}.$

5. $\displaystyle\int x(a^2-x^2)^{n+1/2}\,dx = -\frac{1}{2n+3}\,(a^2-x^2)^{n+3/2}.$

6. $\displaystyle\int(a^2-x^2)^{n+1/2}\,dx = \frac{x}{2n+2}\,(a^2-x^2)^{n+1/2}+\frac{(2n+1)a^2}{2n+2}\int(a^2-x^2)^{n-1/2}\,dx.$

7. $\displaystyle\qquad = \frac{x(a^2-x^2)^{1/2}}{2n+2}\times$

$$\times\left[(a^2-x^2)^n+\sum_{k=1}^{n}\frac{(2n+1)(2n-1)\dots(2n-2k+3)}{2^k n(n-1)\dots(n-k+1)}\,a^{2k}(a^2-x^2)^{n-k}\right]+$$

$$+\frac{(2n+1)!!a^{2n+2}}{2^n(n+1)!}\arcsin\frac{x}{|a|}.$$

8. $\displaystyle\int(a^2-x^2)^{1/2}\,dx = \frac{x}{2}\,(a^2-x^2)^{1/2}+\frac{a^2}{2}\arcsin\frac{x}{|a|}.$

9. $\displaystyle\int x(a^2-x^2)^{1/2}\,dx = -\frac{1}{3}\,(a^2-x^2)^{3/2}.$

10. $\displaystyle\int x^2(a^2-x^2)^{1/2}\,dx = \frac{x}{8}\,(2x^2-a^2)(a^2-x^2)^{1/2}+\frac{a^4}{8}\arcsin\frac{x}{|a|}.$

11. $\displaystyle\int x^3(a^2-x^2)^{1/2}\,dx = \frac{1}{5}\,(a^2-x^2)^{5/2}-\frac{a^2}{3}\,(a^2-x^2)^{3/2}.$

12. $\displaystyle\int(a^2-x^2)^{3/2}\,dx = \frac{x}{4}\,(a^2-x^2)^{3/2}+\frac{3a^2x}{8}\,(a^2-x^2)^{1/2}+\frac{3a^4}{8}\arcsin\frac{x}{|a|}.$

13. $\displaystyle\int x(a^2-x^2)^{3/2}\,dx = -\frac{1}{5}\,(a^2-x^2)^{5/2}.$

14. $\displaystyle\int x^2(a^2-x^2)^{3/2}\,dx = -\frac{x}{48}\,(8x^4-14a^2x^2+3a^4)(a^2-x^2)^{1/2}+\frac{a^6}{16}\arcsin\frac{x}{|a|}.$

15. $\displaystyle\int x^3(a^2-x^2)^{3/2}\,dx = \frac{1}{7}\,(a^2-x^2)^{7/2}-\frac{a^2}{5}\,(a^2-x^2)^{5/2}.$

### 2.53. Интегралы вида $\displaystyle\int\frac{(a^2-x^2)^{n+1/2}}{x^m}\,dx.$

1. $\displaystyle\int\frac{(a^2-x^2)^{n+1/2}}{x^m}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{(m-1)a^2x^{m-1}}+$

$$+\frac{m-2n-4}{(m-1)a^2}\int\frac{(a^2-x^2)^{n+1/2}}{x^{m-2}}\,dx.$$

**2.** $$= -\frac{(a^2-x^2)^{n+1/2}}{(m-1)x^{m-1}} - \frac{2n+1}{m-1}\int\frac{(a^2-x^2)^{n-1/2}}{x^{m-2}}\,dx.$$

**3.** $$\int\frac{(a^2-x^2)^{n+1/2}}{x^{2m}}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{(2m-1)a^2}\times$$
$$\times\left[\frac{1}{x^{2m-1}} + \sum_{k=1}^{m-1}\frac{2^k(m-n-2)(m-n-3)\ldots(m-n-k-1)}{(2m-3)(2m-5)\ldots(2m-2k-1)}\,\frac{a^{-2k}}{x^{2m-2k-1}}\right]+$$
$$+\frac{2^m(m-n-2)(m-n-3)\ldots(-n)(-n-1)}{(2m-1)!!\,a^{2m}}\int(a^2-x^2)^{n+1/2}\,dx.$$

**4.** $$\int\frac{(a^2-x^2)^{n+1/2}}{x}\,dx = \sum_{k=0}^{n}\frac{a^{2k}}{2n-2k+1}\,(a^2-x^2)^{n-k+1/2}+$$
$$+\frac{a^{2n+1}}{2}\ln\left|\frac{a-(a^2-x^2)^{1/2}}{a+(a^2-x^2)^{1/2}}\right|.$$

**5.** $$\int\frac{(a^2-x^2)^{n+1/2}}{x^{2n+2}}\,dx =$$
$$=\sum_{k=0}^{n}\frac{(-1)^{k+1}}{2n-2k+1}\left(\frac{a^2-x^2}{x^2}\right)^{n-k+1/2} + (-1)^{n+1}\arcsin\frac{x}{|a|}.$$

**6.** $$\int\frac{(a^2-x^2)^{n+1/2}}{x^{2n+4}}\,dx = -\frac{(a^2-x^2)^{n+3/2}}{(2n+3)a^2x^{2n+3}}.$$

**7.** $$\int\frac{(a^2-x^2)^{1/2}}{x}\,dx = (a^2-x^2)^{1/2} - a\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**8.** $$\int\frac{(a^2-x^2)^{1/2}}{x^2}\,dx = -\frac{(a^2-x^2)^{1/2}}{x} - \arcsin\frac{x}{|a|}.$$

**9.** $$\int\frac{(a^2-x^2)^{1/2}}{x^3}\,dx = -\frac{(a^2-x^2)^{1/2}}{2x^2} + \frac{1}{2a}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**10.** $$\int\frac{(a^2-x^2)^{3/2}}{x}\,dx =$$
$$=\frac{1}{3}\,(a^2-x^2)^{3/2} + a^2(a^2-x^2)^{1/2} - a^3\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**11.** $$\int\frac{(a^2-x^2)^{3/2}}{x^2}\,dx = -\frac{1}{x}\,(a^2-x^2)^{3/2} - \frac{3x}{2}\,(a^2-x^2)^{1/2} - \frac{3a^2}{2}\arcsin\frac{x}{|a|}.$$

**12.** $$\int\frac{(a^2-x^2)^{3/2}}{x^3}\,dx =$$
$$=-\frac{(a^2-x^2)^{3/2}}{2x^2} - \frac{3}{2}\,(a^2-x^2)^{1/2} + \frac{3a}{2}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

### 2.54. Интегралы вида $\int\frac{x^m\,dx}{(a^2-x^2)^{n+1/2}}$.

**1.** $$\int\frac{x^m\,dx}{(a^2-x^2)^{n+1/2}} = \frac{x^{m-1}}{(2n-1)(a^2-x^2)^{n-1/2}} - \frac{m-1}{2n-1}\int\frac{x^{m-2}\,dx}{(a^2-x^2)^{n-1/2}}.$$

**2.** $$= \frac{x^{m+1}}{(2n-1)a^2(a^2-x^2)^{n-1/2}} - \frac{m-2n+2}{(2n-1)a^2}\int\frac{x^m\,dx}{(a^2-x^2)^{n-1/2}}.$$

**3.** $$= -\frac{x^{m-1}}{(m-2n)(a^2-x^2)^{n-1/2}} + \frac{(m-1)a^2}{m-2n}\int \frac{x^{m-2}\,dx}{(a^2-x^2)^{n+1/2}}.$$

**4.** $$\int \frac{x^{2m+1}\,dx}{(a^2-x^2)^{n+1/2}} = \sum_{k=0}^{m} \frac{(-1)^k a^{2m-2k}}{2n-2k-1} C_m^k \frac{1}{(a^2-x^2)^{n-k-1/2}}.$$

**5.** $$\int \frac{x^{2m}\,dx}{(a^2-x^2)^{n+1/2}} = \frac{x^{2m+1}(a^2-x^2)^{1/2}}{(2n-1)a^2}\left[\frac{1}{(a^2-x^2)^n} + \right.$$
$$\left. + \sum_{k=1}^{n-1} \frac{2^k(n-m-1)(n-m-2)\ldots(n-m-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \frac{a^{2k}}{(a^2-x^2)^{n-k}}\right] +$$
$$+ \frac{2^n(n-m-1)(n-m-2)\ldots(-m+1)(-m)}{(2n-1)!!\,a^{2n}} \int \frac{x^{2m}\,dx}{(a^2-x^2)^{1/2}}.$$

**6.** $$= \frac{1}{a^{2n-2m}} \sum_{k=0}^{n-m-1} \frac{C_{n-m-1}^k}{2m+2k+1}\left(\frac{x^2}{a^2-x^2}\right)^{m+k+1/2} \qquad [n \geqslant m+1].$$

**7.** $$\int \frac{x^{2n-2}\,dx}{(a^2-x^2)^{n+1/2}} = \frac{x^{2n-1}}{(2n-1)a^2(a^2-x^2)^{n-1/2}}.$$

**8.** $$\int \frac{x^{2m}\,dx}{(a^2-x^2)^{1/2}} = -\frac{(a^2-x^2)^{1/2}}{2m} \times$$
$$\times \left[x^{2m-1} + \sum_{k=1}^{m-1} \frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m-1)(m-2)\ldots(m-k)} a^{2k} x^{2m-2k-1}\right] +$$
$$+ \frac{(2m-1)!!}{2^m m!} a^{2m} \arcsin\frac{x}{|a|}.$$

**9.** $$\int \frac{dx}{(a^2-x^2)^{n+1/2}} = \frac{x(a^2-x^2)^{1/2}}{2n-1}\left[\frac{1}{a^2(a^2-x^2)^n} + \right.$$
$$\left. + \sum_{k=1}^{n-1} \frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \cdot \frac{a^{-2k-2}}{(a^2-x^2)^{n-k}}\right] \qquad [n \geqslant 1].$$

**10.** $$= \frac{1}{a^{2n}} \sum_{k=0}^{n-1} \frac{C_{n-1}^k}{2k+1}\left(\frac{x^2}{a^2-x^2}\right)^{k+1/2}.$$

**11.** $$\int \frac{x\,dx}{(a^2-x^2)^{n+1/2}} = \frac{1}{(2n-1)(a^2-x^2)^{n-1/2}}.$$

**12.** $$\int \frac{dx}{(a^2-x^2)^{1/2}} = \arcsin\frac{x}{|a|}.$$

**13.** $$\int \frac{x\,dx}{(a^2-x^2)^{1/2}} = -(a^2-x^2)^{1/2}.$$

**14.** $$\int \frac{x^2\,dx}{(a^2-x^2)^{1/2}} = -\frac{x}{2}(a^2-x^2)^{1/2} + \frac{a^2}{2}\arcsin\frac{x}{|a|}.$$

**15.** $$\int \frac{x^3\,dx}{(a^2-x^2)^{1/2}} = \frac{1}{3}(a^2-x^2)^{3/2} - a^2(a^2-x^2)^{1/2}.$$

**16.** $$\int \frac{dx}{(a^2-x^2)^{3/2}} = \frac{x}{a^2(a^2-x^2)^{1/2}}.$$

**17.** $\displaystyle\int \frac{x\,dx}{(a^2-x^2)^{3/2}} = \frac{1}{(a^2-x^2)^{1/2}}.$

**18.** $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \frac{x}{(a^2-x^2)^{1/2}} - \arcsin\frac{x}{|a|}.$

**19.** $\displaystyle\int \frac{x^3\,dx}{(a^2-x^2)^{3/2}} = \frac{2a^2-x^2}{(a^2-x^2)^{1/2}}.$

### 2.55. Интегралы вида $\displaystyle\int \frac{dx}{x^m(a^2-x^2)^{n+1/2}}$.

**1.** $$\int \frac{dx}{x^m(a^2-x^2)^{n+1/2}} = \frac{1}{(2n-1)a^2x^{m-1}(a^2-x^2)^{n-1/2}} + \frac{m+2n-2}{(2n-1)a^2}\int \frac{dx}{x^m(a^2-x^2)^{n-1/2}}.$$

**2.** $$= -\frac{1}{(m-1)a^2x^{m-1}(a^2-x^2)^{n-1/2}} + \frac{m+2n-2}{(m-1)a^2}\int \frac{dx}{x^{m-2}(a^2-x^2)^{n+1/2}}.$$

**3.** $$= \frac{1}{(2n-1)x^{m+1}(a^2-x^2)^{n-1/2}} + \frac{m+1}{2n-1}\int \frac{dx}{x^{m+2}(a^2-x^2)^{n-1/2}}.$$

**4.** $$\int \frac{dx}{x^{2m}(a^2-x^2)^{n+1/2}} = -\frac{1}{a^{2m+2n}}\sum_{k=0}^{m+n-1}\frac{C_{m+n-1}^k}{2m-2k-1}\left(\frac{a^2-x^2}{x^2}\right)^{m-k-1/2} \quad [m+n \geqslant 1].$$

**5.** $$\int \frac{dx}{x(a^2-x^2)^{n+1/2}} = \frac{1}{(2n-1)a^2(a^2-x^2)^{n-1/2}} + \frac{1}{a^2}\int \frac{dx}{x(a^2-x^2)^{n-1/2}}.$$

**6.** $$= \sum_{k=0}^{n-1}\frac{1}{(2n-2k-1)a^{2k+2}(a^2-x^2)^{n-k-1/2}} - \frac{1}{a^{2k+1}}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**7.** $$\int \frac{dx}{x(a^2-x^2)^{1/2}} = -\frac{1}{a}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**8.** $$\int \frac{dx}{x^2(a^2-x^2)^{1/2}} = -\frac{1}{a^2x}(a^2-x^2)^{1/2}.$$

**9.** $$\int \frac{dx}{x^3(a^2-x^2)^{1/2}} = -\frac{1}{2a^2x^2}(a^2-x^2)^{1/2} - \frac{1}{2a^3}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**10.** $$\int \frac{dx}{x(a^2-x^2)^{3/2}} = \frac{1}{a^2(a^2-x^2)^{1/2}} - \frac{1}{a^3}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**11.** $$\int \frac{dx}{x^2(a^2-x^2)^{3/2}} = \frac{2x^2-a^2}{a^4x(a^2-x^2)^{1/2}}.$$

**12.** $$\int \frac{dx}{x^3(a^2-x^2)^{3/2}} = -\frac{1}{2a^2x^2(a^2-x^2)^{1/2}} + \frac{3}{2a^4(a^2-x^2)^{1/2}} - \frac{3}{2a^5}\ln\left|\frac{a+(a^2-x^2)^{1/2}}{x}\right|.$$

**2.56.** Интегралы вида $\displaystyle\int \frac{dx}{(x+b)^n(a^2-x^2)^{1/2}}$.

**1.** $$\int \frac{dx}{(x+b)^n(a^2-x^2)^{1/2}} = \frac{1}{(n-1)(b^2-a^2)}\left[\frac{(a^2-x^2)^{1/2}}{(x+b)^{n-1}} + {}\right.$$
$$\left. + (2n-3)b\int \frac{dx}{(x+b)^{n-1}(a^2-x^2)^{1/2}} - (n-2)\int \frac{dx}{(x+b)^{n-2}(a^2-x^2)^{1/2}}\right]$$
$$[b \neq \pm a].$$

**2.** $$= -\frac{1}{(2n-1)a}\left[\frac{(a^2-x^2)^{1/2}}{(x+a)^n} - (n-1)\int \frac{dx}{(x+a)^{n-1}(a^2-x^2)^{1/2}}\right] \qquad [b = a].$$

**3.** $$= \frac{1}{(2n-1)a}\left[\frac{(a^2-x^2)^{1/2}}{(x-a)^n} - (n-1)\int \frac{dx}{(x-a)^{n-1}(a^2-x^2)^{1/2}}\right] \qquad [b = -a].$$

**4.** $$\int \frac{dx}{(x+b)(a^2-x^2)^{1/2}} =$$
$$= -\frac{1}{(a^2-b^2)^{1/2}}\ln\left|\frac{bx+a^2+(a^2-b^2)^{1/2}(a^2-x^2)^{1/2}}{x+b}\right| \qquad [a^2 > b^2].$$

**5.** $$= -\frac{1}{(b^2-a^2)^{1/2}}\operatorname{arctg}\frac{(b^2-a^2)^{1/2}(a^2-x^2)^{1/2}}{a^2+bx} \qquad [a^2 < b^2].$$

**6.** $$= \frac{1}{(b^2-a^2)^{1/2}}\arcsin\frac{a^2+bx}{a|x+b|} \qquad [a^2 < b^2].$$

**7.** $$\int \frac{dx}{(x+a)(a^2-x^2)^{1/2}} = -\frac{1}{a}\left(\frac{a-x}{a+x}\right)^{1/2} \qquad [a > 0].$$

**8.** $$\int \frac{dx}{(x-a)(a^2-x^2)^{1/2}} = -\frac{1}{a}\left(\frac{a+x}{a-x}\right)^{1/2} \qquad [a > 0].$$

**2.57.** Интегралы вида $\displaystyle\int \frac{dx}{(b^2 \pm x^2)^m(a^2-x^2)^{n+1/2}}$.

**1.** $$\int \frac{dx}{(b^2-x^2)^m(a^2-x^2)^{n+1/2}} =$$
$$= \frac{x}{2(m-1)(a^2-b^2)b^2(b^2-x^2)^{m-1}(a^2-x^2)^{n-1/2}} + {}$$
$$+ \frac{(2m-3)a^2-2(2m+n-3)b^2}{2(m-1)(a^2-b^2)b^2}\int \frac{dx}{(b^2-x^2)^{m-1}(a^2-x^2)^{n+1/2}} + {}$$
$$+ \frac{m+n-2}{(m-1)(a^2-b^2)b^2}\int \frac{dx}{(b^2-x^2)^{m-2}(a^2-x^2)^{n+1/2}}.$$

**2.** $$\int \frac{dx}{(b^2-x^2)(a^2-x^2)^{n+1/2}} = -\frac{x}{(2n-1)(a^2-b^2)a^2(a^2-x^2)^{n-1/2}} + {}$$
$$+ \frac{(4n-3)a^2-2(n-1)b^2}{(2n-1)(a^2-b^2)a^2}\int \frac{dx}{(b^2-x^2)(a^2-x^2)^{n-1/2}} - {}$$
$$- \frac{2(n-1)}{(2n-1)(a^2-b^2)a^2}\int \frac{dx}{(b^2-x^2)(a^2-x^2)^{n-3/2}}.$$

**3.** $$\int \frac{dx}{(b^2-x^2)(a^2-x^2)^{1/2}} = \frac{1}{b(a^2-b^2)^{1/2}} \ln \frac{|x(a^2-b^2)^{1/2}+b(a^2-x^2)^{1/2}|}{(x^2-b^2)^{1/2}} \qquad [a^2>b^2].$$

**4.** $$= \frac{1}{b(b^2-a^2)^{1/2}} \operatorname{arctg} \frac{x(b^2-a^2)^{1/2}}{b(a^2-x^2)^{1/2}} \qquad [a^2<b^2].$$

**5.** $$= \frac{1}{b(b^2-a^2)^{1/2}} \arcsin \frac{x(b^2-a^2)^{1/2}}{a(b^2-x^2)^{1/2}} \qquad [a^2<b^2].$$

**6.** $$\int \frac{dx}{(b^2+x^2)(a^2-x^2)^{1/2}} = \frac{1}{b(a^2+b^2)^{1/2}} \operatorname{arctg} \frac{x(a^2+b^2)^{1/2}}{b(a^2-x^2)^{1/2}}.$$

### 2.58. Интегралы вида $\int x^m(ax^2+bx+c)^{n+1/2}\,dx$.

Обозначение: $X = ax^2+bx+c$.

**1.** $$\int x^m X^{n+1/2}\,dx = \frac{x^{m-1}X^{n+3/2}}{(m+2n+2)a} - \frac{(2m+2n+1)b}{2(m+2n+2)a}\int x^{m-1}X^{n+1/2}\,dx - \frac{(m-1)c}{(m+2n+2)a}\int x^{m-2}X^{n+1/2}\,dx.$$

**2.** $$\int xX^{n+1/2}\,dx = \frac{X^{n+3/2}}{(2n+3)a} - \frac{b}{2a}\int X^{n+1/2}\,dx.$$

**3.** $$\int X^{n+1/2}\,dx = \frac{2ax+b}{4(n+1)a}X^{n+1/2} + \frac{(2n+1)(4ac-b^2)}{8(n+1)a}\int X^{n-1/2}\,dx.$$

**4.** $$= \frac{(2ax+b)X^{1/2}}{4(n+1)a}\left[X^n + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{8^{k+1}n(n-1)\ldots(n-k)} \times \left(\frac{4ac-b^2}{a}\right)^{k+1} X^{n-k-1}\right] + \frac{(2n+1)!!}{8^{n+1}(n+1)!}\left(\frac{4ac-b^2}{a}\right)^{n+1}\int \frac{dx}{X^{1/2}}.$$

**5.** $$\int x^m X^{1/2}\,dx = \frac{x^{m-1}X^{3/2}}{(m+2)a} - \frac{(2m+1)b}{2(m+2)a}\int x^{m-1}X^{1/2}\,dx - \frac{(m-1)c}{(m+2)a}\int x^{m-2}X^{1/2}\,dx.$$

**6.** $$\int X^{1/2}\,dx = \frac{2ax+b}{4a}X^{1/2} + \frac{4ac-b^2}{8a^{3/2}}\ln\left|\frac{2ax+b}{2\sqrt{a}} + X^{1/2}\right| \qquad [a>0].$$

**7.** $$= \frac{2ax+b}{4a}X^{1/2} + \frac{b^2-4ac}{8a\sqrt{|a|}}\arcsin\frac{2ax+b}{\sqrt{b^2-4ac}} \qquad [a<0;\ b^2>4ac].$$

**8.** $$\int xX^{1/2}\,dx = \frac{1}{3a}X^{3/2} - \frac{b}{2a}\int X^{1/2}\,dx.$$

**9.** $$\int x^2X^{1/2}\,dx = \frac{6ax-5b}{24a^2}X^{3/2} + \frac{5b^2-4ac}{16a^2}\int X^{1/2}\,dx.$$

**10.** $$\int x^3X^{1/2}\,dx = \frac{48a^2x^2-42abx+35b^2-32ac}{240a^2}X^{3/2} - \frac{7b^3-12abc}{32a^3}\int X^{1/2}\,dx.$$

**2.59.** Интегралы вида $\displaystyle\int \frac{(ax^2+bx+c)^{n+1/2}}{x^m}\,dx$.

Обозначение: $X = ax^2 + bx + c$.

**1.** $$\int \frac{X^{n+1/2}}{x^m}\,dx = -\frac{X^{n+3/2}}{(m-1)cx^{m-1}} + \frac{(2n-2m+5)b}{2(m-1)c}\int \frac{X^{n+1/2}}{x^{m-1}}\,dx + \frac{(2n-m+4)a}{(m-1)c}\int \frac{X^{n+1/2}}{x^{m-2}}\,dx.$$

**2.** $$\int \frac{X^{n+1/2}}{x}\,dx = \frac{X^{n+1/2}}{2n+1} + \frac{b}{2}\int X^{n-1/2}\,dx + c\int \frac{X^{n-1/2}}{x}\,dx.$$

**3.** $$\int \frac{(ax^2+bx)^{n+1/2}}{x^m}\,dx = \frac{2(ax^2+bx)^{n+3/2}}{(2n-2m+3)bx^m} + \frac{2(m-2n-3)a}{(2n-2m+3)b}\int \frac{(ax^2+bx)^{n+1/2}}{x^{m-1}}\,dx.$$

**4.** $$\int \frac{X^{1/2}}{x}\,dx = X^{1/2} + \frac{b}{2}\int \frac{dx}{X^{1/2}} + c\int \frac{dx}{xX^{1/2}}.$$

**5.** $$\int \frac{X^{1/2}}{x^2}\,dx = -\frac{X^{1/2}}{x} + a\int \frac{dx}{X^{1/2}} + \frac{b}{2}\int \frac{dx}{xX^{1/2}}.$$

**6.** $$\int \frac{(ax^2+bx)^{1/2}}{x^2}\,dx = -\frac{2(ax^2+bx)^{1/2}}{x} + a\int \frac{dx}{(ax^2+bx)^{1/2}}.$$

**7.** $$\int \frac{X^{1/2}\,dx}{x^3} = -\left(\frac{1}{2x^2} + \frac{b}{4cx}\right)X^{1/2} + \left(\frac{a}{2} - \frac{b^2}{8c}\right)\int \frac{dx}{xX^{1/2}}.$$

**8.** $$\int \frac{(ax^2+bx)^{1/2}}{x^3}\,dx = -\frac{2(ax^2+bx)^{3/2}}{3bx^3}.$$

**9.** $$\int \frac{X^{3/2}}{x}\,dx = \frac{X^{3/2}}{3} + \frac{2abx+b^2+8ac}{8a}X^{1/2} + \frac{(12ac-b^2)b}{16a}\int \frac{dx}{X^{1/2}} + c^2\int \frac{dx}{xX^{1/2}}.$$

**10.** $$\int \frac{X^{3/2}}{x^2}\,dx = -\frac{X^{5/2}}{cx} + \frac{ax+b}{c}X^{3/2} + \frac{3(2ax+3b)}{4}X^{1/2} + \frac{3(4ac+b^2)}{8}\int \frac{dx}{X^{1/2}} + \frac{3bc}{2}\int \frac{dx}{xX^{1/2}}.$$

**11.** $$\int \frac{(ax^2+bx)^{3/2}}{x^2}\,dx = \frac{(ax^2+bx)^{3/2}}{2x} + \frac{3b}{4}(ax^2+bx)^{1/2} + \frac{3b^2}{8}\int \frac{dx}{(ax^2+bx)^{1/2}}.$$

**12.** $$\int \frac{X^{3/2}}{x^3}\,dx = -\left(\frac{1}{2cx^2} + \frac{b}{4c^2x}\right)X^{5/2} + \frac{abx+2ac+b^2}{4c^2}X^{3/2} + \frac{3(abx+2ac+b^2)}{4c}X^{1/2} + \frac{3ab}{2}\int \frac{dx}{X^{1/2}} + \frac{3(4ac+b^2)}{8}\int \frac{dx}{xX^{1/2}}.$$

**13.** $$\int \frac{(ax^2+bx)^{3/2}}{x^3}\,dx = \left(a - \frac{2b}{x}\right)(ax^2+bx)^{1/2} + \frac{3ab}{2}\int \frac{dx}{(ax^2+bx)^{1/2}}.$$

**2.60. Интегралы вида** $\displaystyle\int \frac{dx}{(ax^2+bx+c)^{n+1/2}}$.

Обозначение: $X = ax^2 + bx + c$.

1. $$\int \frac{dx}{X^{n+1/2}} = \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} + \frac{8(n-1)a}{(2n-1)(4ac-b^2)} \int \frac{dx}{X^{n-1/2}}.$$

2. $$= \frac{2(2ax+b)}{(2n-1)(4ac-b^2)X^{n-1/2}} \times \\ \times \left[1 + \sum_{k=1}^{n-1} \frac{8^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \left(\frac{aX}{4ac-b^2}\right)^k\right].$$

3. $$\int \frac{dx}{X^{1/2}} = \frac{1}{\sqrt{a}} \ln\left|\frac{2ax+b}{2\sqrt{a}} + X\right| \qquad [a > 0].$$

4. $$= \frac{1}{\sqrt{a}}\ \operatorname{Arsh} \frac{2ax+b}{\sqrt{4ac-b^2}} \qquad [a > 0;\ b^2 < 4ac].$$

5. $$= -\frac{1}{\sqrt{-a}} \arcsin \frac{2ax+b}{\sqrt{b^2-4ac}} \qquad [a < 0;\ b^2 > 4ac].$$

6. $$= \frac{1}{\sqrt{a}} \ln(2ax+b) \qquad [a > 0;\ b^2 = 4ac].$$

7. $$\int \frac{dx}{X^{3/2}} = \frac{2(2ax+b)}{(4ac-b^2)X^{1/2}}.$$

**2.61. Интегралы вида** $\displaystyle\int \frac{x^m\,dx}{(ax^2+bx+c)^{n+1/2}}$.

Обозначение: $X = ax^2 + bx + c$.

1. $$\int \frac{x^m\,dx}{X^{n+1/2}} = \frac{x^{m-1}}{(m-2n)aX^{n-1/2}} - \frac{(2m-2n-1)b}{2(m-2n)a} \int \frac{x^{m-1}\,dx}{X^{n+1/2}} - \\ - \frac{(m-1)c}{(m-2n)a} \int \frac{x^{m-2}\,dx}{X^{n+1/2}}.$$

2. $$= \frac{2x^m}{(2n-1)bX^{n-1/2}} + \frac{2(2n-m-1)}{(2n-1)b} \int \frac{x^{m-1}\,dx}{X^{n-1/2}} - \frac{2c}{b} \int \frac{x^{m-1}\,dx}{X^{n+1/2}}.$$

3. $$= \frac{1}{a} \int \frac{x^{m-2}\,dx}{X^{n-1/2}} - \frac{c}{a} \int \frac{x^{m-2}\,dx}{X^{n+1/2}} - \frac{b}{a} \int \frac{x^{m-1}\,dx}{X^{n+1/2}}.$$

4. $$\int \frac{x^{2n}\,dx}{X^{n+1/2}} = -\frac{x^{2n-1}}{(2n-1)aX^{n-1/2}} - \frac{b}{2a} \int \frac{x^{2n-1}\,dx}{X^{n+1/2}} + \frac{1}{a} \int \frac{x^{2n-2}\,dx}{X^{n-1/2}}.$$

5. $$\int \frac{x\,dx}{X^{n+1/2}} = -\frac{1}{(2n-1)aX^{n-1/2}} - \frac{b}{2a} \int \frac{dx}{X^{n+1/2}}.$$

6. $$\int \frac{x\,dx}{X^{1/2}} = \frac{X^{1/2}}{a} - \frac{b}{2a} \int \frac{dx}{X^{1/2}}.$$

7. $$\int \frac{x^2\,dx}{X^{1/2}} = \frac{2ax-3b}{4a^2} X^{1/2} + \frac{3b^2-4ac}{8a^2} \int \frac{dx}{X^{1/2}}.$$

8. $$\int \frac{x^3\,dx}{X^{1/2}} = \frac{8a^2x^2-10abx+15b^2-16ac}{24a^3} X^{1/2} - \frac{5b^3-12abc}{16a^3} \int \frac{dx}{X^{1/2}}.$$

9. $$\int \frac{x\,dx}{X^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)X^{1/2}}.$$

10. $\displaystyle\int \frac{x^2\,dx}{X^{3/2}} = \frac{(4ac-2b^2)x-2bc}{a(b^2-4ac)X^{1/2}} + \frac{1}{a}\int \frac{dx}{X^{1/2}}.$

11. $\displaystyle\int \frac{x^3\,dx}{X^{3/2}} = \frac{a(4ac-b^2)x^2+b(10ac-3b^2)x+c(8ac-3b^2)}{a^2(4ac-b^2)X^{1/2}} - \frac{3b}{2a^2}\int \frac{dx}{X^{1/2}}.$

**2.62.** Интегралы вида $\displaystyle\int \frac{dx}{x^m(ax^2+bx+c)^{n+1/2}}$.

Обозначение: $X = ax^2+bx+c$.

1. $$\int \frac{dx}{x^m X^{n+1/2}} = -\frac{1}{(m-1)cx^{m-1}X^{n-1/2}} - \\ - \frac{(2m+2n-3)b}{2(m-1)c}\int \frac{dx}{x^{m-1}X^{n+1/2}} - \frac{(m+2n-2)a}{(m-1)c}\int \frac{dx}{x^{m-2}X^{n+1/2}}.$$

2. $\displaystyle\int \frac{dx}{xX^{n+1/2}} = \frac{1}{(2n-1)cX^{n-1/2}} - \frac{b}{2c}\int \frac{dx}{X^{n+1/2}} + \frac{1}{c}\int \frac{dx}{xX^{n-1/2}}.$

3. $$\int \frac{dx}{x^m(ax^2+bx)^{n+1/2}} = -\frac{2}{(2m+2n-1)bx^m(ax^2+bx)^{n-1/2}} - \\ - \frac{2(m+2n-1)a}{(2m+2n-1)b}\int \frac{dx}{x^{m-1}(ax^2+bx)^{n+1/2}}.$$

4. $\displaystyle\int \frac{dx}{xX^{1/2}} = -\frac{1}{\sqrt{c}}\ln\left|\frac{2c+bx+2\sqrt{c}\,X^{1/2}}{x}\right|$ $[c>0]$.

5. $\displaystyle\qquad = -\frac{1}{\sqrt{c}}\ \operatorname{Arth}\frac{2c+bx}{2\sqrt{c}\,X^{1/2}}$ $[c>0]$.

6. $\displaystyle\qquad = -\frac{1}{\sqrt{c}}\ \operatorname{Arsh}\frac{2c+bx}{x\sqrt{4ac-b^2}}$ $[c>0;\ b^2<4ac]$.

7. $\displaystyle\qquad = \frac{1}{\sqrt{c}}\ln\left|\frac{x}{2c+bx}\right|$ $[c>0;\ b^2=4ac]$.

8. $\displaystyle\qquad = \frac{1}{\sqrt{-c}}\operatorname{arctg}\frac{2c+bx}{2\sqrt{-c}\,X^{1/2}}$ $[c<0]$.

9. $\displaystyle\qquad = \frac{1}{\sqrt{-c}}\arcsin\frac{2c+bx}{x\sqrt{b^2-4ac}}$ $[c<0;\ b^2>4ac]$.

10. $\displaystyle\qquad = -\frac{2\sqrt{bx+ax^2}}{bx}$ $[c=0;\ b\neq 0]$.

11. $\displaystyle\int \frac{dx}{x^2X^{1/2}} = -\frac{X^{1/2}}{cx} - \frac{b}{2c}\int \frac{dx}{xX^{1/2}}.$

12. $\displaystyle\int \frac{dx}{x^2(ax^2+bx)^{1/2}} = \frac{2}{3}\left(-\frac{1}{bx^2} + \frac{2a}{b^2x}\right)(ax^2+bx)^{1/2}.$

13. $\displaystyle\int \frac{dx}{x^3X^{1/2}} = \left(-\frac{1}{2cx^2} + \frac{3b}{4c^2x}\right)X^{1/2} + \left(\frac{3b^2}{8c^2} - \frac{a}{2c}\right)\int \frac{dx}{xX^{1/2}}.$

14. $\displaystyle\int \frac{dx}{x^3(ax^2+bx)^{1/2}} = \frac{2}{5}\left(-\frac{1}{bx^3} + \frac{4a}{3b^2x^2} - \frac{8a^2}{3b^3x}\right)(ax^2+bx)^{1/2}.$

15. $\displaystyle\int \frac{dx}{xX^{3/2}} = \frac{2(abx-2ac+b^2)}{c(b^2-4ac)X^{1/2}} + \frac{1}{c}\int \frac{dx}{xX^{1/2}}.$

16. $$\int\frac{dx}{x(ax^2+bx)^{3/2}}=\frac{2}{3}\left(-\frac{1}{bx}+\frac{4a}{b^2}-\frac{8a^2x}{b^3}\right)\frac{1}{(ax^2+bx)^{1/2}}.$$

17. $$\int\frac{dx}{x^2X^{3/2}}=\left[-\frac{1}{cx}-\frac{b(10ac-3b^2)}{c^2(4ac-b^2)}-\frac{a(8ac-3b^2)x}{c^2(4ac-b^2)}\right]\frac{1}{X^{1/2}}-\frac{3b}{2c^2}\int\frac{dx}{xX^{1/2}}.$$

18. $$\int\frac{dx}{x^2(ax^2+bx)^{3/2}}=\frac{2}{5}\left(-\frac{1}{bx^2}+\frac{2a}{b^2x}-\frac{8a^2}{b^3}-\frac{16a^3x}{b^4}\right)\frac{1}{(ax^2+bx)^{1/2}}.$$

19. $$\int\frac{dx}{x^3X^{3/2}}=\left[-\frac{1}{cx^2}+\frac{5b}{2c^2x}+\frac{15b^4-62acb^2+24a^2c^2}{2c^3(b^2-4ac)}+\right.$$
$$\left.+\frac{ab(15b^2-52ac)x}{2c^3(b^2-4ac)}\right]\frac{1}{2X^{1/2}}+\frac{15b^2-12ac}{8c^3}\int\frac{dx}{xX^{1/2}}.$$

20. $$\int\frac{dx}{x^3(ax^2+bx)^{3/2}}=$$
$$=\frac{2}{7}\left(-\frac{1}{bx^3}+\frac{8a}{5b^2x^2}-\frac{16a^2}{5b^3x}+\frac{64a^3}{5b^4}+\frac{128a^4x}{5b^5}\right)\frac{1}{(ax^2+bx)^{1/2}}.$$

**2.63.** Интегралы вида $\int\frac{(ax^2+bx+c)^{\pm n+1/2}}{(x+p)^m}\,dx$.

Обозначение: $X=ax^2+bx+c$.

1. $$\int\frac{X^{n+1/2}}{(x+p)^m}\,dx=$$
$$=-\frac{X^{n+3/2}}{(m-1)(ap^2-bp+c)(x+p)^{m-1}}+\frac{(2n-2m+5)(b-2ap)}{2(m-1)(ap^2-bp+c)}\times$$
$$\times\int\frac{X^{n+1/2}}{(x+p)^{m-1}}\,dx+\frac{(2n-m+4)a}{(m-1)(ap^2-bp+c)}\int\frac{X^{n+1/2}}{(x+p)^{m-2}}\,dx.$$

2. $$=\frac{2X^{n+3/2}}{(2n-2m+3)(b-2ap)(x+p)^m}-\frac{2(2n-m+3)a}{(2n-2m+3)(b-2ap)}\times$$
$$\times\int\frac{X^{n+1/2}}{(x+p)^{m-1}}\,dx\qquad[ap^2-bp+c=0;\ b-2ap\neq0].$$

3. $$\int\frac{X^{n+1/2}}{x+p}\,dx=$$
$$=\frac{X^{n+1/2}}{2n+1}+\frac{b-2ap}{2}\int X^{n-1/2}\,dx+(ap^2-bp+c)\int\frac{X^{n-1/2}}{x+p}\,dx.$$

4. $$\int\frac{X^{1/2}\,dx}{(x+p)^m}=-\frac{X^{1/2}}{(m-1)(x+p)^{m-1}}+\frac{1}{2(m-1)}\int\frac{(b+2ax)\,dx}{(x+p)^{m-1}X^{1/2}}.$$

5. $$\int\frac{X^{1/2}\,dx}{x+p}=-\int\frac{ax+b-ap}{X^{1/2}}\,dx+(ap^2-bp+c)\int\frac{dx}{(x+p)X^{1/2}}.$$

6. $$\int\frac{dx}{(x+p)^mX^{n+1/2}}=-\frac{1}{(m-1)(ap^2-bp+c)(x+p)^{m-1}X^{n-1/2}}-$$
$$-\frac{(2m+2n-3)(b-2ap)}{2(m-1)(ap^2-bp+c)}\int\frac{dx}{(x+p)^{m-1}X^{n+1/2}}-$$
$$-\frac{(m+2n-2)a}{(m-1)(ap^2-bp+c)}\int\frac{dx}{(x+p)^{m-2}X^{n+1/2}}.$$

**7.** $$= -\frac{2}{(2m+2n-1)(b-2ap)(x+p)^m X^{n-1/2}} - \frac{2(m+2n-1)a}{(2m+2n-1)(b-2ap)} \int \frac{dx}{(x+p)^{m-1} X^{n+1/2}}$$ $$\left[ap^2 - bp + c = 0;\ b - 2ap \neq 0\right].$$

**8.** $$\int \frac{dx}{(x+p)X^{n+1/2}} = \frac{1}{(2n-1)(ap^2-bp+c)X^{n-1/2}} + \frac{1}{ap^2-bp+c} \int \frac{dx}{(x+p)X^{n-1/2}} + \frac{2ap-b}{2(ap^2-bp+c)} \int \frac{dx}{X^{n+1/2}}.$$

**9.** $$= -\frac{1}{(2n+1)X^{n+1/2}} + \frac{b-2ap}{2} \int \frac{dx}{X^{n+3/2}} \qquad \left[ap^2 - bp + c = 0\right].$$

**10.** $$\int \frac{dx}{(x+p)^m X^{1/2}} = -\frac{X^{1/2}}{(m-1)(ap^2-bp+c)(x+p)^{m-1}} - \frac{(2m-3)(b-2ap)}{2(m-1)(ap^2-bp+c)} \times \int \frac{dx}{(x+p)^{m-1} X^{1/2}} - \frac{(m-2)a}{(m-1)(ap^2-bp+c)} \int \frac{dx}{(x+p)^{m-2} X^{1/2}}.$$

**11.** $$= \frac{2X^{1/2}}{(2m-1)(2ap-b)(x+p)^m} + \frac{2(m-1)a}{2(m-1)(2ap-b)} \int \frac{dx}{(x+p)^{m-1} X^{1/2}} \qquad \left[ap^2 - bp + c = 0;\ 2ap - b \neq 0\right].$$

**12.** $$= -\int \frac{t^{m-1}\,dt}{[a + (b-2ap)t + (ap^2-bp+c)t^2]^{1/2}} \qquad \left[t = \frac{1}{x+p}\right].$$

**13.** $$\int \frac{dx}{(x+p)X^{1/2}} = -\frac{1}{(ap^2-bp+c)^{1/2}} \times \ln\left|\frac{(ap^2-bp+c)^{1/2} + X^{1/2}}{x+p} + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}}\right| \qquad \left[ap^2 - bp + c > 0\right].$$

**14.** $$= \frac{1}{2(ap^2-bp+c)^{1/2}} \times \ln\left|\frac{(ap^2-bp+c)^{1/2} - X^{1/2}}{x+p} + \frac{b-2ap}{2(ap^2-bp+c)^{1/2}}\right| \qquad \left[ap^2 - bp + c > 0\right].$$

**15.** $$= \frac{1}{|ap^2-bp+c|^{1/2}} \arcsin \frac{(b-2ap)x - bp + 2c}{(x+p)(b^2-4ac)^{1/2}}$$ $$\left[ap^2 - bp + c < 0;\ b^2 > 4ac\right].$$

**16.** $$= -\frac{2X^{1/2}}{(b-2ap)(x+p)} \qquad \left[ap^2 - bp + c = 0\right].$$

**2.64.** Интегралы вида $\int \frac{dx}{(x^2+px+q)^m(ax^2+bx+c)^{1/2}}$.

Обозначение: $X = ax^2 + bx + c$.

**1.** $$\int \frac{dx}{(x^2+px+q)X^{1/2}} = \frac{1}{(p^2-4q)^{1/2}}\left[\int \frac{dx}{\left(x+\frac{p-\sqrt{p^2-4q}}{2}\right)X^{1/2}} - \int \frac{dx}{\left(x+\frac{p+\sqrt{p^2-4q}}{2}\right)X^{1/2}}\right]$$

$[p^2 > 4q;$ см. 2.63$]$.

**2.** $$\int \frac{dx}{(x^2+p^2)X^{1/2}} = \int \frac{(\alpha-\beta)(t+1)\,dt}{[(\alpha^2+p^2)t^2+(\beta^2+p^2)][(a\alpha^2+b\alpha+c)t^2+\alpha\beta^2+b\beta+c]^{1/2}}$$

$\left[t = \frac{x-\beta}{\alpha-x};\right.$ $\alpha$ и $\beta$ определяются из системы уравнений $\left. b(\alpha+\beta)+2c-2ap^2=0,\ \alpha\beta+p^2=0\right]$.

**3.** $$\int \frac{(\alpha x+\beta)\,dx}{(p+X)^m X^{1/2}} = \frac{\alpha}{a}\int \frac{du}{(p+u^2)^m} + \frac{2\beta a-\alpha b}{2a}\int \frac{(1-av^2)^{m-1}\,dv}{\left(p+c-\frac{b^2}{4a}-apv^2\right)^m}$$

$\left[u = X^{1/2};\ v = \frac{b+2ax}{2aX^{1/2}};\right.$ см. 2.15, 2.17$\left.\right]$.

## 3. ПОКАЗАТЕЛЬНАЯ ФУНКЦИЯ

### 3.1. Интегралы вида $\int f(e^{ax})\,dx$.

**1.** $\displaystyle\int f(e^{ax})\,dx = \frac{1}{a}\int \frac{f(t)}{t}\,dt \qquad [t = e^{ax}].$

**2.** $\displaystyle\int f(b^x)\,dx = \int f(e^{x\ln b})\,dx \qquad [b > 0,\ b \neq 1].$

**3.** $\displaystyle\int e^{ax}\,dx = \frac{1}{a}\,e^{ax}.$

**4.** $\displaystyle\int b^x\,dx = \frac{b^x}{\ln b} \qquad [b > 0,\ b \neq 1].$

**5.** $\displaystyle\int \frac{dx}{e^{ax}+b} = \frac{1}{ab}\,[ax - \ln(e^{ax}+b)].$

**6.** $\displaystyle\int \frac{dx}{be^{ax}+ce^{-ax}} = \frac{1}{a\sqrt{bc}}\,\operatorname{arctg}\left(e^{ax}\sqrt{\frac{b}{c}}\right) \qquad [bc > 0].$

**7.** $\displaystyle\qquad = \frac{1}{2a\sqrt{-bc}}\ln\frac{c+e^{ax}\sqrt{-bc}}{c-e^{ax}\sqrt{-bc}} \qquad [bc < 0].$

**8.** $\displaystyle\int \frac{e^{ax}-1}{e^{ax}+1}\,dx = \frac{2}{a}\ln\operatorname{ch}\frac{ax}{2}.$

**9.** $\displaystyle\int \frac{dx}{\sqrt{b+ce^{ax}}} = \frac{1}{a\sqrt{b}}\ln\frac{\sqrt{b+ce^{ax}}-\sqrt{b}}{\sqrt{b+ce^{ax}}+\sqrt{b}} \qquad [b > 0].$

**10.** $\displaystyle\qquad = \frac{2}{a\sqrt{-b}}\operatorname{arctg}\frac{\sqrt{b+ce^{ax}}}{\sqrt{-b}} \qquad [b < 0].$

### 3.2. Интегралы вида $\int x^p e^{ax}\,dx$.

**1.** $\displaystyle\int x^p e^{ax}\,dx = \frac{1}{a}\,x^p e^{ax} - \frac{p}{a}\int x^{p-1}e^{ax}\,dx.$

**2.** $$\int x^{p-n-1}e^{-ax}\,dx = \frac{(-1)^n a^{n-1}}{(n-p)(n-p-1)\dots(1-p)}\int x^{p-1}e^{-ax}\,dx - \\ - \frac{e^{-ax}}{x^{n-p}}\sum_{k=0}^{n-1}\frac{(-1)^k(ax)^k}{(n-p)(n-p-1)\dots(1-p-k)}.$$

**3.** $\displaystyle\int x^n e^{ax}\,dx = e^{ax}\left(\frac{x^n}{a} + \sum_{k=1}^{n}(-1)^k\frac{n(n-1)\dots(n-k+1)}{a^{k+1}}\,x^{n-k}\right).$

**4.** $\displaystyle\int xe^{ax}\,dx = e^{ax}\left(\frac{x}{a} - \frac{1}{a^2}\right).$

**5.** $\displaystyle\int x^2 e^{ax}\,dx = e^{ax}\left(\frac{x^2}{a} - \frac{2x}{a^2} + \frac{2}{a^3}\right).$

**6.** $\displaystyle\int x^3 e^{ax}\,dx = e^{ax}\left(\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right).$

**7.** $\displaystyle\int P(x)e^{ax}\,dx = \frac{e^{ax}}{a}\sum_{k=0}^{m}\frac{(-1)^k}{a^k}\,\frac{d^k}{dx^k}\,P(x),$

где $P(x)$ — многочлен степени $m$.

**3.3. Интегралы вида** $\displaystyle\int \frac{e^{ax}}{x^p}\,dx.$

**1.** $\displaystyle\int \frac{e^{ax}}{x^p}\,dx = -\frac{e^{ax}}{(p-1)x^{p-1}} + \frac{a}{p-1}\int\frac{e^{ax}}{x^{p-1}}\,dx.$

**2.** $\displaystyle\int \frac{e^{ax}}{x^n}\,dx = -e^{ax}\sum_{k=1}^{n-1}\frac{a^{k-1}}{(n-1)(n-2)\ldots(n-k)x^{n-k}} + \frac{a^{n-1}}{(n-1)!}\,\mathrm{Ei}\,(ax).$

**3.** $\displaystyle\int \frac{e^{ax}}{x}\,dx = \mathrm{Ei}\,(ax)$ $\qquad [a \neq 0].$

**4.** $\displaystyle\qquad = \ln|x| + \sum_{k=1}^{\infty}\frac{(ax)^k}{k!k}.$

**5.** $\displaystyle\int \frac{e^{-ax}}{x}\,dx = -\,\mathrm{Ei}\,(-ax)$ $\qquad [a > 0].$

**6.** $\displaystyle\int \frac{e^{-ax}}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{a}}\;\mathrm{erf}\,(\sqrt{ax}\,)$ $\qquad [a > 0].$

**7.** $\displaystyle\int \frac{e^{-ax}}{x^{3/2}}\,dx = -\frac{2}{\sqrt{x}}\,e^{-ax} - 2\sqrt{\pi a}\;\mathrm{erf}\,(\sqrt{ax}\,)$ $\qquad [a > 0].$

**3.4. Интегралы вида** $\displaystyle\int x^p e^{-ax^2}\,dx.$

**1.** $\displaystyle\int e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\;\mathrm{erf}\,(\sqrt{a}\;x).$

**2.** $\displaystyle\int x^p e^{-ax^2}\,dx = -\frac{x^{p-1}}{2a}\,e^{-ax^2} + \frac{p-1}{2a}\int x^{p-2}e^{-ax^2}\,dx.$

**3.** $$\int x^{2n} e^{-ax^2}\,dx = -\frac{(2n-1)!!\,x^{2n-1}e^{-ax^2}}{2a}\times$$
$$\times\sum_{k=0}^{n-1}\frac{1}{2^k(2n-2k-1)!!\,(ax^2)^k} + \frac{(2n-1)!!\,\sqrt{\pi}}{2^{n+1}a^{(2n+1)/2}}\,\mathrm{erf}\,(\sqrt{a}\;x).$$

**4.** $\displaystyle\int x^{2n+1} e^{-ax^2}\,dx = -\frac{x^{2n}e^{-ax^2}}{2a}\sum_{k=0}^{n}\frac{n!}{(n-k)!a^k x^{2k}}.$

**5.** $\displaystyle\int x e^{-ax^2}\,dx = -\frac{1}{2a}\,e^{-ax^2}.$

**6.** $\displaystyle\int x^2 e^{-ax^2}\,dx = -\frac{x}{2a}e^{-ax^2} + \frac{\sqrt{\pi}}{4a^{3/2}}\operatorname{erf}(\sqrt{a}\ x).$

**7.** $\displaystyle\int x^3 e^{-ax^2}\,dx = -\frac{ax^2+1}{2a^2}\,e^{-ax^2}.$

**3.5. Интегралы вида** $\displaystyle\int \frac{e^{-ax^2}}{x^p}\,dx.$

**1.** $\displaystyle\int \frac{e^{-ax^2}}{x^p}\,dx = -\frac{e^{-ax^2}}{(p-1)x^{p-1}} - \frac{2a}{p-1}\int \frac{e^{-ax^2}}{x^{p-2}}\,dx.$

**2.** $$\int \frac{1}{x^{2n}}e^{-ax^2}\,dx = \frac{e^{-ax^2}}{(2n-1)!!\,2a}\sum_{k=1}^{n}(-2)^k(2n-2k-1)!!\,a^k x^{2k-2n-1} - \frac{(-2)^{n-1}\sqrt{\pi}\ a^{(2n-1)/2}}{(2n-1)!!}\operatorname{erf}(\sqrt{a}\ x).$$

**3.** $$\int \frac{1}{x^{2n+1}}e^{-ax^2}\,dx = \frac{1}{n!\,2a}e^{-ax^2}\sum_{k=1}^{n}(-1)^k(n-k)!\,a^k x^{2k-2n-2} + \frac{(-a)^n}{n!\,2}\operatorname{Ei}(-ax^2).$$

**4.** $\displaystyle\int \frac{e^{-ax^2}}{x}\,dx = \frac{1}{2}\operatorname{Ei}(-ax^2).$

**5.** $\displaystyle\int \frac{e^{-ax^2}}{x^2}\,dx = -\frac{1}{x}e^{-ax^2} - \sqrt{\pi a}\ \operatorname{erf}(\sqrt{a}\ x).$

**6.** $\displaystyle\int \frac{e^{-ax^2}}{x^3}\,dx = -\frac{1}{2x^2}e^{-ax^2} - \frac{a}{2}\operatorname{Ei}(-ax^2).$

**7.** $$\int f(x)e^{-(ax^2+bx+c)}\,dx = \frac{1}{\sqrt{a}}\,e^{(b^2-4ac)/(4a)}\int f\left(\frac{2\sqrt{a}\,t-b}{2a}\right)e^{-t^2}\,dt \qquad \left[t=\sqrt{a}\left(x+\frac{b}{2a}\right),\ a>0\right].$$

**8.** $\displaystyle\int e^{-(ax^2+bx+c)}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}}\,e^{(b^2-4ac)/(4a)}\operatorname{erf}\left(\sqrt{a}\ x+\frac{b}{2\sqrt{a}}\right).$

## 4. ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

### 4.1. Интегралы вида $\int \operatorname{sh}^p x\,dx$.

**1.** $$\int \operatorname{sh}^p x\,dx = \frac{1}{p}\operatorname{sh}^{p-1} x \operatorname{ch} x - \frac{p-1}{p}\int \operatorname{sh}^{p-2} x\,dx.$$

**2.** $$\int \operatorname{sh}^{2m} x\,dx = (-1)^m C_{2m}^m \frac{x}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\frac{(-1)^k C_{2m}^k}{2m-2k}\operatorname{sh}(2m-2k)x.$$

**3.** $$\int \operatorname{sh}^{2m+1} x\,dx = \sum_{k=0}^{m}(-1)^{m+k}\frac{C_m^k}{2k+1}\operatorname{ch}^{2k+1} x.$$

**4.** $$= \frac{1}{2^{2m}}\sum_{k=0}^{m}\frac{(-1)^k C_{2m+1}^k}{2m-2k+1}\operatorname{ch}(2m-2k+1)x.$$

**5.** $$\int \operatorname{sh} x\,dx = \operatorname{ch} x.$$

**6.** $$\int \operatorname{sh}^2 x\,dx = \frac{1}{4}\operatorname{sh} 2x - \frac{x}{2}.$$

**7.** $$\int \operatorname{sh}^3 x\,dx = -\frac{3}{4}\operatorname{ch} x + \frac{1}{12}\operatorname{ch} 3x = -\operatorname{ch} x + \frac{1}{3}\operatorname{ch}^3 3x.$$

**8.** $$\int \operatorname{sh}^4 x\,dx = \frac{3}{8}x - \frac{1}{4}\operatorname{sh} 2x + \frac{1}{32}\operatorname{sh} 4x = \frac{3}{8}x - \frac{3}{8}\operatorname{sh} x \operatorname{ch} x + \frac{1}{4}\operatorname{sh}^3 x \operatorname{ch} x.$$

### 4.2. Интегралы вида $\int \frac{dx}{\operatorname{sh}^p x}$.

**1.** $$\int \frac{dx}{\operatorname{sh}^p x}\,dx = -\frac{\operatorname{ch} x}{(p-1)\operatorname{sh}^{p-1} x} - \frac{p-2}{p-1}\int \frac{dx}{\operatorname{sh}^{p-2} x}.$$

**2.** $$\int \frac{dx}{\operatorname{sh}^{2m} x} = \frac{\operatorname{ch} x}{2m-1}\left[-\frac{1}{\operatorname{sh}^{2m-1} x} + \sum_{k=1}^{m-1}(-1)^{k-1}\frac{2^k(m-1)(m-2)\ldots(m-k)}{(2m-3)(2m-5)\ldots(2m-2k-1)}\operatorname{sh}^{2k-2m+1} x\right].$$

**3.** $$\int \frac{dx}{\operatorname{sh}^{2m+1} x} = \frac{\operatorname{ch} x}{2m}\left[-\frac{1}{\operatorname{sh}^{2m} x} + \sum_{k=1}^{m-1}(-1)^{k-1}\frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m-1)(m-2)\ldots(m-k)}\operatorname{sh}^{2k-2m} x\right] + (-1)^m\frac{(2m-1)!!}{(2m)!!}\ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**4.** $$\int \frac{dx}{\operatorname{sh} x} = \ln\left|\operatorname{th}\frac{x}{2}\right| = \frac{1}{2}\ln\frac{\operatorname{ch} x - 1}{\operatorname{ch} x + 1}.$$

5. $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x.$

6. $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x} = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} - \frac{1}{2}\ln\left|\operatorname{th}\frac{x}{2}\right|.$

7. $\displaystyle\int \frac{dx}{\operatorname{sh}^4 x} = -\frac{1}{3}\operatorname{cth}^3 x + \operatorname{cth} x.$

**4.3.** Интегралы вида $\displaystyle\int \operatorname{ch}^p x\,dx.$

1. $\displaystyle\int \operatorname{ch}^p x\,dx = \frac{1}{p}\operatorname{sh} x\operatorname{ch}^{p-1} x - \frac{p-1}{p}\int \operatorname{ch}^{p-2} x\,dx.$

2. $\displaystyle\int \operatorname{ch}^{2m} x\,dx = C_{2m}^m\frac{x}{2^{2m}} + \frac{1}{2^{2m-1}}\sum_{k=0}^{m-1}\frac{C_{2m}^k}{2m-2k}\operatorname{ch}(2m-2k)x.$

3. $\displaystyle\int \operatorname{ch}^{2m+1} x\,dx = \sum_{k=0}^{m}\frac{C_m^k}{2k+1}\operatorname{sh}^{2k+1} x.$

4. $\displaystyle\qquad = \frac{1}{2^{2m}}\sum_{k=0}^{m}\frac{C_{2m+1}^k}{2m-2k+1}\operatorname{sh}(2m-2k+1)x.$

5. $\displaystyle\int \operatorname{ch} x\,dx = \operatorname{sh} x.$

6. $\displaystyle\int \operatorname{ch}^2 x\,dx = \frac{1}{4}\operatorname{sh} 2x + \frac{x}{2}.$

7. $\displaystyle\int \operatorname{ch}^3 x\,dx = \frac{3}{4}\operatorname{sh} x + \frac{1}{12}\operatorname{sh} 3x = \operatorname{sh} x + \frac{1}{3}\operatorname{sh}^3 x.$

8. $\displaystyle\int \operatorname{ch}^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\operatorname{sh} 2x + \frac{1}{32}\operatorname{sh} 4x = \frac{3}{8}x + \frac{3}{8}\operatorname{sh} x\operatorname{ch} x + \frac{1}{4}\operatorname{sh} x\operatorname{ch}^3 x.$

**4.4.** Интегралы вида $\displaystyle\int \frac{dx}{\operatorname{ch}^p x}.$

1. $\displaystyle\int \frac{dx}{\operatorname{ch}^p x}\,dx = \frac{\operatorname{sh} x}{(p-1)\operatorname{ch}^{p-1} x} + \frac{p-2}{p-1}\int \frac{dx}{\operatorname{ch}^{p-2} x}.$

2. $$\int \frac{dx}{\operatorname{ch}^{2m} x} = \frac{\operatorname{sh} x}{2m-1}\left[\frac{1}{\operatorname{ch}^{2m-1} x} + \right.$$
$$\left. + \sum_{k=1}^{m-1}\frac{2^k(m-1)(m-2)\ldots(m-k)}{(2m-3)(2m-5)\ldots(2m-2k-1)}\operatorname{ch}^{2k-2m+1} x\right].$$

3. $$\int \frac{dx}{\operatorname{ch}^{2m+1} x} = \frac{\operatorname{sh} x}{2m}\left[\frac{1}{\operatorname{ch}^{2m} x} + \sum_{k=1}^{m-1}\frac{(2m-1)(2m-3)\ldots(2m-2k+1)}{2^k(m-1)(m-2)\ldots(m-k)}\times\right.$$
$$\left.\times\operatorname{ch}^{2k-2m} x\right] + \frac{(2m-1)!!}{(2m)!!}\operatorname{arctg}\operatorname{sh} x.$$

4. $\displaystyle\int \frac{dx}{\operatorname{ch} x} = \operatorname{arctg}\operatorname{sh} x = 2\operatorname{arctg} e^x.$

5. $\displaystyle\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x.$

**6.** $\displaystyle\int \frac{dx}{\operatorname{ch}^3 x} = \frac{\operatorname{sh} x}{2\operatorname{ch}^2 x} + \frac{1}{2}\operatorname{arctg}\operatorname{sh} x.$

**7.** $\displaystyle\int \frac{dx}{\operatorname{ch}^4 x} = -\frac{1}{3}\operatorname{th}^3 x + \operatorname{th} x.$

## 4.5. Интегралы вида $\int \operatorname{sh}^p x \operatorname{ch}^q x\,dx$.

**1.** $\displaystyle\int \operatorname{sh}^p \operatorname{ch}^q x\,dx = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \operatorname{sh}^p x \operatorname{ch}^{q-2} x\,dx.$

**2.** $\displaystyle\qquad = \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{p+q} - \frac{p-1}{p+q}\int \operatorname{sh}^{p-2} x \operatorname{ch}^{q} x\,dx.$

**3.** $\displaystyle\qquad = \frac{\operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x}{q+1} - \frac{p-1}{q+1}\int \operatorname{sh}^{p-2} x \operatorname{ch}^{q+2} x\,dx.$

**4.** $\displaystyle\qquad = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x}{p+1} - \frac{q-1}{p+1}\int \operatorname{sh}^{p+2} x \operatorname{ch}^{q-2} x\,dx.$

**5.** $\displaystyle\qquad = \frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{p+1} - \frac{p+q+2}{p+1}\int \operatorname{sh}^{p+2} x \operatorname{ch}^{q} x\,dx.$

**6.** $\displaystyle\qquad = -\frac{\operatorname{sh}^{p+1} x \operatorname{ch}^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \operatorname{sh}^{p} x \operatorname{ch}^{q+2} x\,dx.$

**7.** $$\int \operatorname{sh}^p x \operatorname{ch}^{2n} x\,dx = \frac{\operatorname{sh}^{p+1} x}{2n+p}\left[\operatorname{ch}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)}\operatorname{ch}^{2n-2k-1} x\right] + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)}\int \operatorname{sh}^p x\,dx \qquad [p \neq -2,\ -4,\ \ldots,\ -2n].$$

**8.** $$\int \operatorname{sh}^p x \operatorname{ch}^{2n+1} x\,dx = \frac{\operatorname{sh}^{p+1} x}{2n+p+1}\left[\operatorname{ch}^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\ldots(n-k+1)}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)}\operatorname{ch}^{2n-2k} x\right] \qquad [p \neq -1,\ -3,\ \ldots,\ -2n-1].$$

**9.** $$\int \operatorname{sh}^{2n} x \operatorname{ch}^{p} x\,dx = \frac{\operatorname{ch}^{p+1} x}{2n+p}\left[\operatorname{sh}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(-1)^k(2n-1)(2n-3)\ldots(2n-2k+1)}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)}\operatorname{sh}^{2n-2k-1} x\right] + \frac{(-1)^n(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)}\int \operatorname{ch}^p x\,dx \qquad [p \neq -2,\ -4,\ \ldots,\ -2n].$$

**10.** $$\int \operatorname{sh}^{2n+1} x \operatorname{ch}^{p} x\,dx = \frac{\operatorname{ch}^{p+1} x}{2n+p+1}\left[\operatorname{sh}^{2n} x + \sum_{k=1}^{n} \frac{(-2)^k n(n-1)\ldots(n-k+1)}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)}\operatorname{sh}^{2n-2k} x\right] \qquad [p \neq -1,\ -3,\ \ldots,\ -2n-1].$$

11. $\int \operatorname{sh} ax \operatorname{ch} bx\, dx = \dfrac{\operatorname{ch}(a+b)x}{2(a+b)} + \dfrac{\operatorname{ch}(a-b)x}{2(a-b)}.$

12. $\int \operatorname{sh} x \operatorname{ch} x\, dx = \dfrac{1}{4}\operatorname{ch} 2x.$

13. $\int \operatorname{sh} x \operatorname{ch}^p x\, dx = \dfrac{1}{p+1}\operatorname{ch}^{p+1} x.$

14. $\int \operatorname{sh}^2 x \operatorname{ch}^2 x\, dx = -\dfrac{x}{8} + \dfrac{1}{32}\operatorname{sh} 4x.$

15. $\int \operatorname{sh}^2 x \operatorname{ch}^3 x\, dx = \dfrac{1}{5}\operatorname{sh}^3 x \operatorname{ch}^2 x + \dfrac{2}{15}\operatorname{sh}^3 x.$

16. $\int \operatorname{sh}^2 x \operatorname{ch}^4 x\, dx = -\dfrac{x}{16} - \dfrac{1}{64}\operatorname{sh} 2x + \dfrac{1}{64}\operatorname{sh} 4x + \dfrac{1}{192}\operatorname{sh} 6x.$

17. $\int \operatorname{sh}^3 x \operatorname{ch}^2 x\, dx = \dfrac{1}{5}\operatorname{sh}^2 x \operatorname{ch}^3 x - \dfrac{2}{15}\operatorname{ch}^3 x.$

18. $\int \operatorname{sh}^3 x \operatorname{ch}^3 x\, dx = \dfrac{1}{6}\operatorname{sh}^6 x + \dfrac{1}{4}\operatorname{sh}^4 x = \dfrac{1}{6}\operatorname{ch}^6 x - \dfrac{1}{4}\operatorname{ch}^4 x.$

19. $\int \operatorname{sh}^3 x \operatorname{ch}^4 x\, dx = \dfrac{1}{7}\operatorname{sh}^2 x - \dfrac{2}{35}\operatorname{sh}^2 x \operatorname{ch}^5 x.$

20. $\int \operatorname{sh}^4 x \operatorname{ch}^2 x\, dx = \dfrac{x}{16} - \dfrac{1}{64}\operatorname{sh} 2x - \dfrac{1}{64}\operatorname{sh} 4x + \dfrac{1}{192}\operatorname{sh} 6x.$

21. $\int \operatorname{sh}^4 x \operatorname{ch}^3 x\, dx = \dfrac{1}{7}\operatorname{ch}^2 x + \dfrac{2}{35}\operatorname{sh}^5 x \operatorname{ch}^2 x.$

22. $\int \operatorname{sh}^4 x \operatorname{ch}^4 x\, dx = \dfrac{3x}{128} - \dfrac{1}{128}\operatorname{sh} 4x + \dfrac{1}{1024}\operatorname{sh} 8x.$

23. $$\int \operatorname{sh} ax \operatorname{sh} bx \operatorname{sh} cx\, dx = \frac{\operatorname{ch}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{ch}(-a+b+c)x}{4(-a+b+c)} - \\ - \frac{\operatorname{ch}(a-b+c)x}{4(a-b+c)} - \frac{\operatorname{ch}(a+b-c)x}{4(a+b-c)}.$$

24. $$\int \operatorname{sh} ax \operatorname{sh} bx \operatorname{ch} cx\, dx = \frac{\operatorname{sh}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{sh}(-a+b+c)x}{4(-a+b+c)} - \\ - \frac{\operatorname{sh}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh}(a+b-c)x}{4(a+b-c)}.$$

25. $$\int \operatorname{sh} ax \operatorname{ch} bx \operatorname{ch} cx\, dx = \frac{\operatorname{ch}(a+b+c)x}{4(a+b+c)} - \frac{\operatorname{ch}(-a+b+c)x}{4(-a+b+c)} + \\ + \frac{\operatorname{ch}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{ch}(a+b-c)x}{4(a+b-c)}.$$

26. $$\int \operatorname{ch} ax \operatorname{ch} bx \operatorname{ch} cx\, dx = \frac{\operatorname{sh}(a+b+c)x}{4(a+b+c)} + \frac{\operatorname{sh}(-a+b+c)x}{4(-a+b+c)} + \\ + \frac{\operatorname{sh}(a-b+c)x}{4(a-b+c)} + \frac{\operatorname{sh}(a+b-c)x}{4(a+b-c)}.$$

### 4.6. Интегралы вида $\int \dfrac{\operatorname{sh}^p x}{\operatorname{ch}^q x}\, dx$.

1. $\int \dfrac{\operatorname{sh}^p x}{\operatorname{ch}^q x}\, dx = \dfrac{1}{p-q}\dfrac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} - \dfrac{p-1}{p-q}\int \dfrac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^q x}\, dx.$

**2.** $$= \frac{1}{q-1}\frac{\operatorname{sh}^{p+1} x}{\operatorname{ch}^{q-1} x} - \frac{p-q+2}{q-1}\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{q-2} x}\,dx.$$

**3.** $$= -\frac{1}{q-1}\frac{\operatorname{sh}^{p-1} x}{\operatorname{ch}^{q-1} x} + \frac{p-1}{q-1}\int \frac{\operatorname{sh}^{p-2} x}{\operatorname{ch}^{q-2} x}\,dx.$$

**4.** $$\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{2n} x}\,dx = \frac{\operatorname{sh}^{p+1} x}{2n-1}\left[\frac{1}{\operatorname{ch}^{2n-1} x} + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)} \operatorname{ch}^{2k-2n+1} x\right] + \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \operatorname{sh}^p x\,dx.$$

**5.** $$\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{2n+1} x}\,dx = \frac{\operatorname{sh}^{p+1} x}{2n}\left[\frac{1}{\operatorname{ch}^{2n} x} + \sum_{k=1}^{n-1} \frac{(-1)^k(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \operatorname{ch}^{2k-2n} x\right] + \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!}\int \frac{\operatorname{sh}^p x}{\operatorname{ch} x}\,dx.$$

**6.** $$\int \frac{\operatorname{sh}^{2m} x}{\operatorname{ch} x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k-1}\operatorname{sh}^{2k-1} x + (-1)^m \operatorname{arctg}\operatorname{sh} x.$$

**7.** $$\int \frac{\operatorname{sh}^{2m+1} x}{\operatorname{ch} x}\,dx = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k}\operatorname{sh}^{2k} x + (-1)^m \ln\operatorname{ch} x = \sum_{k=1}^{m} \frac{(-1)^{m+k}}{2k} C_m^k \operatorname{ch}^{2k} x + (-1)^m \ln\operatorname{ch} x.$$

**8.** $$\int \frac{\operatorname{sh}^{2m+1} x}{\operatorname{ch}^{2n} x}\,dx = \sum_{k=0}^{m} \frac{(-1)^{m+k} C_m^k}{2k-2n+1}\operatorname{ch}^{2k-2n+1} x.$$

**9.** $$\int \frac{\operatorname{sh}^{2m+1} x}{\operatorname{ch}^{2n+1} x}\,dx = \sum_{\substack{k=0\\k\neq n}}^{m} \frac{(-1)^{m+k}}{2k-2n} C_m^k \operatorname{ch}^{2k-2n} x + (-1)^{m+n} C_m^n \ln\operatorname{ch} x \qquad [n \leqslant m].$$

**10.** $$= \sum_{k=0}^{m} \frac{(-1)^{m+k}}{2k-2n} C_m^k \operatorname{ch}^{2k-2n} x \qquad [n > m].$$

**11.** $$\int \frac{\operatorname{sh} x}{\operatorname{ch}^p x}\,dx = -\frac{1}{(p-1)\operatorname{ch}^{p-1} x}.$$

**12.** $$\int \frac{\operatorname{sh}^p x}{\operatorname{ch}^{p+2} x}\,dx = \frac{1}{p+1}\operatorname{th}^{p+1} x.$$

**13.** $$\int \frac{\operatorname{sh} x}{\operatorname{ch} x}\,dx = \ln\operatorname{ch} x.$$

**14.** $$\int \frac{\operatorname{sh}^2 x}{\operatorname{ch} x}\,dx = \operatorname{sh} x - \operatorname{arctg}\operatorname{sh} x.$$

**15.** $\displaystyle\int \frac{\operatorname{sh}^3 x}{\operatorname{ch} x}\,dx = \frac{1}{2}\operatorname{ch}^2 x - \ln\operatorname{ch} x.$

**16.** $\displaystyle\int \frac{\operatorname{sh}^4 x}{\operatorname{ch} x}\,dx = \frac{1}{3}\operatorname{sh}^3 x - \operatorname{sh} x + \operatorname{arctg}\operatorname{sh} x.$

**17.** $\displaystyle\int \frac{\operatorname{sh} x}{\operatorname{ch}^2 x}\,dx = -\frac{1}{\operatorname{ch} x}.$

**18.** $\displaystyle\int \frac{\operatorname{sh}^2 x}{\operatorname{ch}^2 x}\,dx = x - \operatorname{th} x.$

**19.** $\displaystyle\int \frac{\operatorname{sh}^3 x}{\operatorname{ch}^2 x}\,dx = \operatorname{ch} x + \frac{1}{\operatorname{ch} x}.$

**20.** $\displaystyle\int \frac{\operatorname{sh}^4 x}{\operatorname{ch}^2 x}\,dx = -\frac{3}{2}x + \frac{1}{4}\operatorname{sh} 2x + \operatorname{th} x.$

**21.** $\displaystyle\int \frac{\operatorname{sh} x}{\operatorname{ch}^3 x}\,dx = \frac{1}{2}\operatorname{th}^2 x.$

**22.** $\displaystyle\int \frac{\operatorname{sh}^2 x}{\operatorname{ch}^3 x}\,dx = -\frac{\operatorname{sh} x}{2\operatorname{ch}^2 x} + \frac{1}{2}\operatorname{arctg}\operatorname{sh} x.$

**23.** $\displaystyle\int \frac{\operatorname{sh}^3 x}{\operatorname{ch}^3 x}\,dx = -\frac{1}{2}\operatorname{th}^2 x + \ln\operatorname{ch} x.$

**24.** $\displaystyle\int \frac{\operatorname{sh}^4 x}{\operatorname{ch}^3 x}\,dx = \operatorname{sh} x + \frac{\operatorname{sh} x}{2\operatorname{ch}^2 x} - \frac{3}{2}\operatorname{arctg}\operatorname{sh} x.$

**25.** $\displaystyle\int \frac{\operatorname{sh} x}{\operatorname{ch}^4 x}\,dx = -\frac{1}{3\operatorname{ch}^3 x}.$

**26.** $\displaystyle\int \frac{\operatorname{sh}^2 x}{\operatorname{ch}^4 x}\,dx = \frac{1}{3}\operatorname{th}^3 x.$

**27.** $\displaystyle\int \frac{\operatorname{sh}^3 x}{\operatorname{ch}^4 x}\,dx = -\frac{1}{\operatorname{ch} x} + \frac{1}{3\operatorname{ch}^3 x}.$

**28.** $\displaystyle\int \frac{\operatorname{sh}^4 x}{\operatorname{ch}^4 x}\,dx = -\frac{1}{3}\operatorname{th}^3 x - \operatorname{th} x + x.$

### 4.7. Интегралы вида $\displaystyle\int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x}\,dx.$

**1.** $\displaystyle\int \frac{\operatorname{ch}^q x}{\operatorname{sh}^p x}\,dx = \frac{1}{q-p}\,\frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{q-p}\int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^p x}\,dx.$

**2.** $\displaystyle\qquad = -\frac{1}{p-1}\,\frac{\operatorname{ch}^{q+1} x}{\operatorname{sh}^{p-1} x} + \frac{q-p+2}{p-1}\int \frac{\operatorname{ch}^{q} x}{\operatorname{sh}^{p-2} x}\,dx.$

**3.** $\displaystyle\qquad = -\frac{1}{p-1}\,\frac{\operatorname{ch}^{q-1} x}{\operatorname{sh}^{p-1} x} + \frac{q-1}{p-1}\int \frac{\operatorname{ch}^{q-2} x}{\operatorname{sh}^{p-2} x}\,dx.$

**4.**
$$\int \frac{\operatorname{ch}^p x}{\operatorname{sh}^{2n} x}\,dx = -\frac{\operatorname{ch}^{p+1} x}{2n-1}\left[\frac{1}{\operatorname{sh}^{2n-1} x} + \right.$$
$$\left. + \sum_{k=1}^{n-1}(-1)^k \frac{(2n-p-2)(2n-p-4)\ldots(2n-p-2k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\operatorname{sh}^{2k-2n+1} x\right] +$$
$$+ (-1)^n \frac{(2n-p-2)(2n-p-4)\ldots(-p+2)(-p)}{(2n-1)!!}\int \operatorname{ch}^p x\,dx.$$

**5.** $$\int \frac{\operatorname{ch}^p x}{\operatorname{sh}^{2n+1} x}\,dx = -\frac{\operatorname{ch}^{p+1} x}{2n}\left[\frac{1}{\operatorname{sh}^{2n} x} + \right.$$
$$\left. + \sum_{k=1}^{n-1} (-1)^k \frac{(2n-p-1)(2n-p-3)\ldots(2n-p-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)} \operatorname{sh}^{2k-2n} x\right] +$$
$$+ (-1)^n \frac{(2n-p-1)(2n-p-3)\ldots(3-p)(1-p)}{2^n n!} \int \frac{\operatorname{ch}^p x}{\operatorname{sh} x}\,dx.$$

**6.** $$\int \frac{\operatorname{ch}^{2m} x}{\operatorname{sh} x}\,dx = \sum_{k=1}^{m} \frac{\operatorname{ch}^{2n-1} x}{2k-1} + \ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**7.** $$\int \frac{\operatorname{ch}^{2m+1} x}{\operatorname{sh} x}\,dx = \sum_{k=1}^{m} \frac{\operatorname{sh}^{2k} x}{2k} + \ln|\operatorname{sh} x| = \sum_{k=1}^{m} \frac{C_m^k}{2k} \operatorname{sh}^{2k} x + \ln|\operatorname{sh} x|.$$

**8.** $$\int \frac{\operatorname{ch}^{2m+1} x}{\operatorname{sh}^{2n} x}\,dx = \sum_{k=0}^{m} \frac{C_m^k}{2k-2n+1} \operatorname{sh}^{2k-2n+1} x.$$

**9.** $$\int \frac{\operatorname{ch}^{2m+1} x}{\operatorname{sh}^{2n+1} x}\,dx = \sum_{\substack{k=0\\k\neq n}}^{m} \frac{C_m^k}{2k-2n} \operatorname{sh}^{2k-2n} x + C_m^n \ln|\operatorname{sh} x| \qquad [n \leqslant m].$$

**10.** $$= \sum_{k=0}^{m} \frac{C_m^k}{2k-2n} \operatorname{sh}^{2k-2n} x \qquad [n > m].$$

**11.** $$\int \frac{\operatorname{ch} x}{\operatorname{sh}^p x}\,dx = -\frac{1}{(p-1)\operatorname{sh}^{p-1} x}.$$

**12.** $$\int \frac{\operatorname{ch}^p x}{\operatorname{sh}^{p+2} x}\,dx = -\frac{1}{p+1} \operatorname{cth}^{p+1} x.$$

**13.** $$\int \frac{\operatorname{ch} x}{\operatorname{sh} x}\,dx = \ln|\operatorname{sh} x|.$$

**14.** $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh} x}\,dx = \operatorname{ch} x + \ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**15.** $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh} x}\,dx = \frac{1}{2}\operatorname{ch}^2 x + \ln|\operatorname{sh} x|.$$

**16.** $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh} x}\,dx = \frac{1}{3}\operatorname{ch}^3 x + \operatorname{ch} x + \ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**17.** $$\int \frac{\operatorname{ch} x}{\operatorname{sh}^2 x}\,dx = -\frac{1}{\operatorname{sh} x}.$$

**18.** $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^2 x}\,dx = x - \operatorname{cth} x.$$

**19.** $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^2 x}\,dx = \operatorname{sh} x - \frac{1}{\operatorname{sh} x}.$$

**20.** $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^2 x}\,dx = \frac{3}{2}x + \frac{1}{4}\operatorname{sh} 2x - \operatorname{cth} x.$$

**21.** $$\int \frac{\operatorname{ch} x}{\operatorname{sh}^3 x}\,dx = -\frac{1}{2}\operatorname{cth}^2 x.$$

**22.** $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^3 x}\,dx = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} + \ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**23.** $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^3 x}\,dx = -\frac{1}{2}\operatorname{cth}^2 x + \ln|\operatorname{sh} x|.$$

**24.** $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^3 x}\,dx = \operatorname{ch} x - \frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} + \frac{3}{2}\ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**25.** $$\int \frac{\operatorname{ch} x}{\operatorname{sh}^4 x}\,dx = -\frac{1}{3\operatorname{sh}^3 x}.$$

**26.** $$\int \frac{\operatorname{ch}^2 x}{\operatorname{sh}^4 x}\,dx = \frac{1}{3}\operatorname{cth}^3 x.$$

**27.** $$\int \frac{\operatorname{ch}^3 x}{\operatorname{sh}^4 x}\,dx = -\frac{1}{\operatorname{sh} x} - \frac{1}{3\operatorname{sh}^3 x}.$$

**28.** $$\int \frac{\operatorname{ch}^4 x}{\operatorname{sh}^4 x}\,dx = -\frac{1}{3}\operatorname{cth}^3 x - \operatorname{cth} x + x.$$

### 4.8. Интегралы вида $\int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x}$.

**1.** $$\int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^q x} = -\frac{1}{(p-1)\operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} - \frac{p+q-2}{p-1}\int \frac{dx}{\operatorname{sh}^{p-2} x \operatorname{ch}^q x}.$$

**2.** $$= \frac{1}{(q-1)\operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x} + \frac{p+q-2}{q-1}\int \frac{dx}{\operatorname{sh}^p x \operatorname{ch}^{q-2} x}.$$

**3.** $$\int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch}^{2n} x} = \sum_{k=0}^{m+n-1} \frac{(-1)^k}{2m-2k-1} C_{m+n-1}^k \operatorname{th}^{2k-2m+1} x.$$

**4.** $$\int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch}^{2n+1} x} =$$
$$= \sum_{\substack{k=0\\k\neq m}}^{m+n} \frac{(-1)^{k+1}}{2m-2k} C_{m+n}^k \operatorname{th}^{2k-2m} x + (-1)^m C_{m+n}^m \ln|\operatorname{th} x|.$$

**5.** $$\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^{2m} x} = \sum_{k=1}^{m} \frac{1}{2m-2k+1} \operatorname{ch}^{2k-2m-1} x + \ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**6.** $$\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^{2m+1} x} = \sum_{k=1}^{m} \frac{1}{2m-2k+2} \operatorname{ch}^{2k-2m-2} x + \ln|\operatorname{th} x|.$$

**7.** $$\int \frac{dx}{\operatorname{sh}^{2m} x \operatorname{ch} x} = \sum_{k=1}^{m} \frac{(-1)^k}{2m-2k+1} \operatorname{sh}^{2k-2m-1} x + (-1)^m \operatorname{arctg} \operatorname{sh} x.$$

**8.** $$\int \frac{dx}{\operatorname{sh}^{2m+1} x \operatorname{ch} x} = \sum_{k=1}^{m} \frac{(-1)^k}{2m-2k+2} \operatorname{sh}^{2k-2m-2} x + (-1)^m \ln|\operatorname{th} x|.$$

**9.** $$\int \frac{dx}{\operatorname{sh} x \operatorname{ch} x} = \ln|\operatorname{th} x|.$$

**10.** $$\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^2 x} = \frac{1}{\operatorname{ch} x} + \ln\left|\operatorname{th}\frac{x}{2}\right|.$$

**11.** $\displaystyle\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^3 x} = -\frac{1}{2}\operatorname{th}^2 x + \ln|\operatorname{th} x|.$

**12.** $\displaystyle\int \frac{dx}{\operatorname{sh} x \operatorname{ch}^4 x} = \frac{1}{\operatorname{ch} x} + \frac{1}{3\operatorname{ch}^3 x} + \ln\left|\operatorname{th}\frac{x}{2}\right|.$

**13.** $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch} x} = -\frac{1}{\operatorname{sh} x} - \operatorname{arctg}\operatorname{sh} x.$

**14.** $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^2 x} = -2\operatorname{cth} 2x.$

**15.** $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^3 x} = -\frac{1}{\operatorname{sh} x} - \frac{\operatorname{sh} x}{2\operatorname{ch}^2 x} - \frac{3}{2}\operatorname{arctg}\operatorname{sh} x.$

**16.** $\displaystyle\int \frac{dx}{\operatorname{sh}^2 x \operatorname{ch}^4 x} = \frac{1}{3\operatorname{sh} x \operatorname{ch}^3 x} - \frac{8}{3}\operatorname{cth} 2x.$

**17.** $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch} x} = -\frac{1}{2}\operatorname{cth}^2 x + \ln|\operatorname{cth} x|.$

**18.** $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^2 x} = -\frac{1}{\operatorname{ch} x} - \frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} - \frac{3}{2}\ln\left|\operatorname{th}\frac{x}{2}\right|.$

**19.** $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^3 x} = \frac{1}{2}\operatorname{th}^2 x - \frac{1}{2}\operatorname{cth}^2 x - 2\ln|\operatorname{th} x|.$

**20.** $\displaystyle\int \frac{dx}{\operatorname{sh}^3 x \operatorname{ch}^4 x} = -\frac{2}{\operatorname{ch} x} - \frac{1}{3\operatorname{ch}^3 x} - \frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} - \frac{5}{2}\ln\left|\operatorname{th}\frac{x}{2}\right|.$

**21.** $\displaystyle\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch} x} = \frac{1}{\operatorname{sh} x} - \frac{1}{3\operatorname{sh}^3 x} + \operatorname{arctg}\operatorname{sh} x.$

**22.** $\displaystyle\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^2 x} = -\frac{1}{3\operatorname{sh}^3 x \operatorname{ch} x} + \frac{8}{3}\operatorname{cth} 2x.$

**23.** $\displaystyle\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^3 x} = \frac{2}{\operatorname{sh} x} - \frac{1}{3\operatorname{sh}^3 x} + \frac{\operatorname{sh} x}{2\operatorname{ch}^2 x} + \frac{5}{2}\operatorname{arctg}\operatorname{sh} x.$

**24.** $\displaystyle\int \frac{dx}{\operatorname{sh}^4 x \operatorname{ch}^4 x} = 8\operatorname{cth} 2x - \frac{8}{3}\operatorname{cth}^3 2x.$

## 4.9. Интегралы вида $\int \operatorname{th}^p x\,dx$.

**1.** $\displaystyle\int \operatorname{th}^p x\,dx = -\frac{1}{p-1}\operatorname{th}^{p-1} x + \int \operatorname{th}^{p-2} x\,dx.$

**2.** $\displaystyle\int \operatorname{th}^{2n} x\,dx = -\sum_{k=1}^{n}\frac{\operatorname{th}^{2n-2k+1} x}{2n-2k+1} + x.$

**3.** $\displaystyle\int \operatorname{th}^{2n+1} x\,dx = -\sum_{k=1}^{n}\frac{\operatorname{th}^{2n-2k+2} x}{2n-2k+2} + \ln\operatorname{ch} x.$

**4.** $\displaystyle\qquad = -\frac{1}{2}\sum_{k=1}^{n}\frac{(-1)^k C_n^k}{k\operatorname{ch}^{2k} x} + \ln\operatorname{ch} x.$

**5.** $\displaystyle\int \operatorname{th} x\,dx = \ln\operatorname{ch} x.$

**6.** $\displaystyle\int \operatorname{th}^2 x\,dx = x - \operatorname{th} x.$

**4.10. Интегралы вида** $\int \operatorname{cth}^p x\,dx$.

**1.** $$\int \operatorname{cth}^p x\,dx = -\frac{1}{p-1}\operatorname{cth}^{p-1} x + \int \operatorname{cth}^{p-2} x\,dx.$$

**2.** $$\int \operatorname{cth}^{2n} x\,dx = -\sum_{k=1}^{n} \frac{\operatorname{cth}^{2n-2k+1} x}{2n-2k+1} + x.$$

**3.** $$\int \operatorname{cth}^{2n+1} x\,dx = -\sum_{k=1}^{n} \frac{\operatorname{cth}^{2n-2k+2} x}{2n-2k+2} + \ln|\operatorname{sh} x|.$$

**4.** $$= -\frac{1}{2}\sum_{k=1}^{n} \frac{C_n^k}{k\operatorname{sh}^{2k} x} + \ln|\operatorname{sh} x|.$$

**5.** $$\int \operatorname{cth} x\,dx = \ln|\operatorname{sh} x|.$$

**6.** $$\int \operatorname{cth}^2 x\,dx = x - \operatorname{cth} x.$$

**4.11. Интегралы вида** $$\int \frac{(A + B\operatorname{ch} x + C\operatorname{sh} x)\,dx}{(a + b\operatorname{ch} x + c\operatorname{sh} x)^n (a_1 + b_1\operatorname{ch} x + c_1\operatorname{sh} x)^p}.$$

**1.** $$\int \frac{A + B\operatorname{ch} x + C\operatorname{sh} x}{(a + b\operatorname{ch} x + c\operatorname{sh} x)^n}\,dx =$$
$$= -\frac{Bc - Cb + (Ac - Ca)\operatorname{ch} x + (Ab - Ba)\operatorname{sh} x}{(n-1)(a^2 - b^2 + c^2)(a + b\operatorname{ch} x + c\operatorname{sh} x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 + c^2)} \times$$
$$\times \int \frac{(n-1)(Aa - Bb + Cc) - (n-2)(Ab - Ba)\operatorname{ch} x - (n-2)(Ac - Ca)\operatorname{sh} x}{(a + b\operatorname{ch} x + c\operatorname{sh} x)^{n-1}}\,dx$$
$$[a^2 + c^2 \neq b^2].$$

**2.** $$= \frac{Bc - Cb - Ca\operatorname{ch} x - Ba\operatorname{sh} x}{(n-1)a(a + b\operatorname{ch} x + c\operatorname{sh} x)^n} +$$
$$+ \left[\frac{A}{a} + \frac{n(Bb - Cc)}{(n-1)a^2}\right](c\operatorname{ch} x + b\operatorname{sh} x)\frac{(n-1)!}{(2n-1)!!} \times$$
$$\times \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!\,a^k} \cdot \frac{1}{(a + b\operatorname{ch} x + c\operatorname{sh} x)^{n-k}} \qquad [a^2 + c^2 = b^2].$$

**3.** $$\int \frac{A + B\operatorname{ch} x + C\operatorname{sh} x}{a + b\operatorname{ch} x + c\operatorname{sh} x}\,dx = \frac{Cb - Bc}{b^2 - c^2}\ln|a + b\operatorname{ch} x + c\operatorname{sh} x| +$$
$$+ \frac{Bb - Cc}{b^2 - c^2}x + \left(A - a\frac{Bb - Cc}{b^2 - c^2}\right)\int \frac{dx}{a + b\operatorname{ch} x + c\operatorname{sh} x} \qquad [b^2 \neq c^2].$$

**4.** $$= \frac{C - B}{2a}e^{-x} + \left[\frac{A}{a} + \frac{(C - B)b}{2a^2}\right]x +$$
$$+ \left[\frac{C + B}{2b} + \frac{A}{a} - \frac{(C - B)b}{2a^2}\right]\ln|a + be^x| \qquad [b = c].$$

**5.** $$= \frac{B+C}{2a}\,e^x + \left[\frac{A}{a} - \frac{(B+C)b}{2a^2}\right]x + \left[\frac{C-B}{2b} - \frac{A}{a} - \frac{(B+C)b}{2a^2}\right]\ln\left|a+be^{-x}\right| \qquad [b=-c].$$

**6.** $$\int \frac{dx}{a+b\operatorname{ch}x + c\operatorname{sh}x} = \frac{2}{\sqrt{b^2-a^2-c^2}}\operatorname{arctg}\frac{(b-a)\operatorname{th}\dfrac{x}{2}+c}{\sqrt{b^2-a^2-c^2}} \qquad [b^2>a^2+c^2;\ a\neq b].$$

**7.** $$= \frac{1}{\sqrt{a^2-b^2+c^2}}\ln\left|\frac{(a-b)\operatorname{th}\dfrac{x}{2} - c + \sqrt{a^2-b^2+c^2}}{(a-b)\operatorname{th}\dfrac{x}{2} - c - \sqrt{a^2-b^2+c^2}}\right| \qquad [b^2<a^2+c^2;\ a\neq b].$$

**8.** $$= \frac{1}{c}\ln\left|a + c\operatorname{th}\frac{x}{2}\right| \qquad [a=b;\ c\neq 0].$$

**9.** $$= \frac{2}{(a-b)\operatorname{th}\dfrac{x}{2}+c} \qquad [b^2=a^2+c^2].$$

**10.** $$\int \frac{dx}{(a\operatorname{ch}x + b\operatorname{sh}x)^n} = \frac{1}{(a^2-b^2)^{n/2}}\int \frac{dx}{\operatorname{ch}^n\left(x+\operatorname{Arth}\dfrac{b}{a}\right)} \qquad [a>|b|].$$

**11.** $$= \frac{1}{(b^2-a^2)^{n/2}}\int \frac{dx}{\operatorname{sh}^n\left(x+\operatorname{Arth}\dfrac{a}{b}\right)} \qquad [b>|a|].$$

**12.** $$\int \frac{A+B\operatorname{ch}x + C\operatorname{sh}x}{a\operatorname{ch}x + b\operatorname{sh}x}\,dx = \frac{A}{\sqrt{a^2-b^2}}\operatorname{arctg}\left|\operatorname{sh}\left(x+\operatorname{Arth}\frac{b}{a}\right)\right| + \frac{1}{a^2-b^2}\left[(Ca-Bb)\ln\operatorname{ch}\left(x+\operatorname{Arth}\frac{b}{a}\right) + (Ba-Cb)x\right] \qquad [a>|b|].$$

**13.** $$= \frac{A}{\sqrt{b^2-a^2}}\ln\left|\operatorname{th}\frac{x+\operatorname{Arth}\dfrac{a}{b}}{2}\right| + \frac{1}{b^2-a^2}\left[(Cb-Ba)x + (Bb-Ca)\ln\left|\operatorname{sh}\left(x+\operatorname{Arth}\frac{a}{b}\right)\right|\right] \qquad [b>|a|].$$

**14.** $$= \frac{1}{a}\left[\frac{B+C}{2}\,x - \frac{B-C}{4}\,e^{-2x} - Ae^{-x}\right] \qquad [a=b].$$

**15.** $$= \frac{1}{a}\left[\frac{B-C}{2}\,x + \frac{B+C}{2}\,e^{2x} + Ae^{x}\right] \qquad [a=-b].$$

**16.** $$\int \frac{dx}{a+b\operatorname{sh}x} = \frac{1}{\sqrt{a^2+b^2}}\ln\left|\frac{a\operatorname{th}\dfrac{x}{2} - b + \sqrt{a^2+b^2}}{a\operatorname{th}\dfrac{x}{2} - b - \sqrt{a^2+b^2}}\right|.$$

**17.** $$= \frac{2}{\sqrt{a^2+b^2}}\operatorname{Arth}\frac{a\operatorname{th}\dfrac{x}{2} - b}{\sqrt{a^2+b^2}}.$$

**18.** $$\int \frac{dx}{a+b\operatorname{ch}x} = \frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{b+a\operatorname{ch}x}{a+b\operatorname{ch}x} \qquad [b^2>a^2;\ x<0].$$

**19.** $$= -\frac{1}{\sqrt{b^2-a^2}} \arcsin \frac{b+a\operatorname{ch}x}{a+b\operatorname{ch}x} \qquad [b^2>a^2;\ x>0].$$

**20.** $$= \frac{1}{\sqrt{a^2-b^2}} \ln \left| \frac{a+b+\sqrt{a^2-b^2}\operatorname{th}\frac{x}{2}}{a+b-\sqrt{a^2-b^2}\operatorname{th}\frac{x}{2}} \right| \qquad [a^2>b^2].$$

**21.** $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{(1+\operatorname{ch}x)^n}\,dx = -\frac{B\operatorname{sh}x}{(n-1)(1+\operatorname{ch}x)^n} - \frac{C}{(n-1)(1+\operatorname{ch}x)^{n-1}} + \\ + \left(\frac{n}{n-1}B+A\right)\frac{(n-1)!}{(2n-1)!!}\operatorname{sh}x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!}(1+\operatorname{ch}x)^{k-n}.$$

**22.** $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{1+\operatorname{ch}x}\,dx = Bx + C\ln(1+\operatorname{ch}x) + (A-B)\frac{\operatorname{ch}x-1}{\operatorname{sh}x}.$$

**23.** $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{(1-\operatorname{ch}x)^n}\,dx = -\frac{B\operatorname{sh}x}{(n-1)(1-\operatorname{ch}x)^n} + \frac{C}{(n-1)(1-\operatorname{ch}x)^{n-1}} + \\ + \left(\frac{n}{n-1}B-A\right)\frac{(n-1)!}{(2n-1)!!}\operatorname{sh}x \sum_{k=0}^{n-1} \frac{(2n-2k-3)!!}{(n-k-1)!}(1-\operatorname{ch}x)^{k-n}.$$

**24.** $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{1-\operatorname{ch}x}\,dx = -Bx - C\ln|1-\operatorname{ch}x| + (A+B)\frac{\operatorname{ch}x+1}{\operatorname{sh}x}.$$

**25.** $$\int \frac{A+B\operatorname{ch}x+C\operatorname{sh}x}{(a_1+b_1\operatorname{ch}x+c_1\operatorname{sh}x)(a_2+b_2\operatorname{ch}x+c_2\operatorname{sh}x)}\,dx = \\ = A_0 \ln\left|\frac{a_1+b_1\operatorname{ch}x+c_1\operatorname{sh}x}{a_2+b_2\operatorname{ch}x+c_2\operatorname{sh}x}\right| + \\ + A_1\int\frac{dx}{a_1+b_1\operatorname{ch}x+c_1\operatorname{sh}x} + A_2\int\frac{dx}{a_2+b_2\operatorname{ch}x+c_2\operatorname{sh}x},$$

где

$$A_0 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ A & B & C \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_1 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} b_1 & c_1 \\ B & C \end{vmatrix} & \begin{vmatrix} c_1 & a_1 \\ C & A \end{vmatrix} & \begin{vmatrix} a_1 & b_1 \\ A & B \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} a_1 & b_1 & c_1 \\ \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} B & A \\ b_2 & a_2 \end{vmatrix} \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 - \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$\left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 \neq \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2\right].$$

**26.** $$\int \frac{A\,\mathrm{ch}^2 x + B\,\mathrm{sh}\,x\,\mathrm{ch}\,x + C\,\mathrm{sh}^2 x}{a\,\mathrm{ch}^2 x + b\,\mathrm{sh}\,x\,\mathrm{ch}\,x + c\,\mathrm{sh}^2 x}\,dx = \frac{1}{b^2-(a+c)^2}\Big\{[Bb-(A+C)(a+c)]x + \\ + \frac{1}{2}[(A+C)b - B(a+c)]\ln|a\,\mathrm{ch}^2 x + b\,\mathrm{sh}\,x\,\mathrm{ch}\,x + c\,\mathrm{sh}^2 x| + \\ + \left[\frac{1}{2}(A-C)b^2 + \frac{1}{2}Bb(a-c) + (Ca-Ac)(a+c)\right] f(x)\Big\},$$

где

$$f(x) = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2c\,\mathrm{th}\,x + b - \sqrt{b^2-4ac}}{2c\,\mathrm{th}\,x + b + \sqrt{b^2-4ac}}\right| \qquad [b^2 > 4ac],$$

$$= \frac{2}{\sqrt{4ac-b^2}}\,\mathrm{arctg}\,\frac{2c\,\mathrm{th}\,x + b}{\sqrt{4ac-b^2}} \qquad [b^2 < 4ac],$$

$$= -\frac{2}{2c\,\mathrm{th}\,x + b} \qquad [b^2 = 4ac].$$

**27.** $$\int \frac{A + B\,\mathrm{ch}\,x + C\,\mathrm{sh}\,x}{\mathrm{sh}\,x\,(a + b\,\mathrm{sh}\,x)}\,dx = \frac{1}{a}\left[A\ln\left|\mathrm{th}\,\frac{x}{2}\right| + b\ln\left|\frac{\mathrm{sh}\,x}{a + b\,\mathrm{sh}\,x}\right| + (Ca - Ab)\int\frac{dx}{a + b\,\mathrm{sh}\,x}\right].$$

**28.** $$\int \frac{A + B\,\mathrm{ch}\,x + C\,\mathrm{sh}\,x}{\mathrm{sh}\,x\,(a + b\,\mathrm{ch}\,x)}\,dx = \frac{1}{a^2-b^2}\left\{(Aa + Bb)\ln\left|\mathrm{th}\,\frac{x}{2}\right| + (Ab - Ba)\ln\left|\frac{a + b\,\mathrm{ch}\,x}{\mathrm{sh}\,x}\right|\right\} + C\int\frac{dx}{a + b\,\mathrm{ch}\,x}.$$

**29.** $$\int \frac{A + B\,\mathrm{ch}\,x + C\,\mathrm{sh}\,x}{\mathrm{sh}\,x\,(1 + \mathrm{ch}\,x)}\,dx = \frac{B + A}{2}\ln\left|\mathrm{th}\,\frac{x}{2}\right| - \frac{A - B}{4}\,\mathrm{th}^2\,\frac{x}{2} + C\,\frac{\mathrm{ch}\,x - 1}{\mathrm{sh}\,x}.$$

**30.** $$\int \frac{A + B\,\mathrm{ch}\,x + C\,\mathrm{sh}\,x}{\mathrm{sh}\,x\,(1 - \mathrm{ch}\,x)}\,dx = \frac{B - A}{2}\ln\left|\mathrm{cth}\,\frac{x}{2}\right| + \frac{A + B}{4}\,\mathrm{cth}^2\,\frac{x}{2} + C\,\frac{\mathrm{ch}\,x + 1}{\mathrm{sh}\,x}.$$

**31.** $$\int \frac{A + B\,\mathrm{ch}\,x + C\,\mathrm{sh}\,x}{\mathrm{ch}\,x\,(a + b\,\mathrm{sh}\,x)}\,dx = \frac{1}{a^2+b^2}\left\{(Aa + Cb)\,\mathrm{arctg}\,\mathrm{sh}\,x + (Ab - Ca)\ln\left|\frac{a + b\,\mathrm{sh}\,x}{\mathrm{ch}\,x}\right|\right\} + B\int\frac{dx}{a + b\,\mathrm{sh}\,x}.$$

**32.** $$\int \frac{A + B\operatorname{ch} x + C\operatorname{sh} x}{\operatorname{ch} x\,(a + b\operatorname{ch} x)}\,dx = \frac{1}{a}\left[A\operatorname{arctg}\operatorname{sh} x - C\ln\left|\frac{a + b\operatorname{ch} x}{\operatorname{ch} x}\right| - (Ab - Ba)\int\frac{dx}{a + b\operatorname{ch} x}\right].$$

**33.** $$\int \frac{dx}{a + b\operatorname{sh}^2 x} = \frac{1}{\sqrt{a(b-a)}}\operatorname{arctg}\left(\sqrt{\frac{b}{a} - 1}\,\operatorname{th} x\right) \qquad [b/a > 1].$$

**34.** $$= \frac{1}{\sqrt{a(a-b)}}\operatorname{Arth}\left(\sqrt{1 - \frac{b}{a}}\,\operatorname{th} x\right)$$

$[0 < b/a < 1]$ или $[b/a < 0;\ \operatorname{sh}^2 x < -a/b]$.

**35.** $$= \frac{1}{\sqrt{a(a-b)}}\operatorname{Arcth}\left(\sqrt{1 - \frac{b}{a}}\,\operatorname{th} x\right) \qquad [b/a < 0;\ \operatorname{sh}^2 x > -a/b].$$

**36.** $$= \frac{1}{a}\operatorname{th} x \qquad [a = b].$$

**37.** $$= \frac{1}{a\sqrt{2}}\operatorname{Arth}(\sqrt{2}\,\operatorname{th} x) \qquad [a = -b;\ \operatorname{sh}^2 x < 1].$$

**38.** $$= \frac{1}{a\sqrt{2}}\operatorname{Arcth}(\sqrt{2}\,\operatorname{th} x) \qquad [a = -b;\ \operatorname{sh}^2 x > 1].$$

**39.** $$\int \frac{dx}{(a + b\operatorname{sh}^2 x)^2} = \frac{1}{2a(b-a)}\left[\frac{b\operatorname{sh} x\operatorname{ch} x}{a + b\operatorname{sh}^2 x} + (b - 2a)\int\frac{dx}{a + b\operatorname{sh}^2 x}\right].$$

**40.** $$\int \frac{dx}{a + b\operatorname{ch}^2 x} = \frac{1}{\sqrt{-a(a+b)}}\operatorname{arctg}\left(\sqrt{-\left(1 + \frac{b}{a}\right)}\,\operatorname{cth} x\right) \qquad [b/a < -1].$$

**41.** $$= \frac{1}{\sqrt{a(a+b)}}\operatorname{Arth}\left(\sqrt{1 + \frac{b}{a}}\,\operatorname{cth} x\right)$$

$[-1 < b/a < 0;\ \operatorname{ch}^2 x > -a/b]$.

**42.** $$= \frac{1}{\sqrt{a(a+b)}}\operatorname{Arcth}\left(\sqrt{1 + \frac{b}{a}}\,\operatorname{cth} x\right)$$

$[b/a > 0]$ или $[-1 < b/a < 0;\ \operatorname{ch}^2 x < -a/b]$.

**43.** $$= \frac{1}{a\sqrt{2}}\operatorname{Arcth}(\sqrt{2}\,\operatorname{cth} x) \qquad [a = b].$$

**44.** $$= \frac{1}{a}\operatorname{cth} x \qquad [a = -b].$$

**45.** $$\int \frac{dx}{(a + b\operatorname{ch}^2 x)^2} = \frac{1}{2a(a+b)}\left[-\frac{b\operatorname{sh} x\operatorname{ch} x}{a + b\operatorname{ch}^2 x} + (2a + b)\int\frac{dx}{a + b\operatorname{ch}^2 x}\right].$$

**46.** $$\int \frac{dx}{a + b\operatorname{th}^2 x} = \frac{1}{a+b}\left[x + \sqrt{\frac{b}{a}}\operatorname{arctg}\left(\sqrt{\frac{b}{a}}\,\operatorname{th} x\right)\right] \qquad [ab > 0;\ a + b \neq 0].$$

**47.** $$= \frac{1}{a+b}\left[x + \frac{b}{2\sqrt{-ab}}\ln\left|\frac{\sqrt{-ab} - b\operatorname{th} x}{\sqrt{-ab} + b\operatorname{th} x}\right|\right] \qquad [ab < 0;\ a + b \neq 0].$$

**48.** $$= \frac{1}{2a}x + \frac{1}{4a}\operatorname{sh} 2x \qquad [a + b = 0].$$

**4.12. Интегралы вида** $\int \operatorname{sh}(ax+b)\operatorname{sh}(cx+d)\,dx$, $\int \operatorname{ch}(ax+b)\operatorname{ch}(cx+d)\,dx$, $\int \operatorname{sh}(ax+b)\operatorname{ch}(cx+d)\,dx$.

**1.** $$\int \operatorname{sh}(ax+b)\operatorname{sh}(cx+d)\,dx = \frac{1}{2(a+c)}\operatorname{sh}[(a+c)x+b+d] - \frac{1}{2(a-c)}\operatorname{sh}[(a-c)x+b-d].$$

**2.** $$\int \operatorname{ch}(ax+b)\operatorname{ch}(cx+d)\,dx = \frac{1}{2(a+c)}\operatorname{sh}[(a+c)x+b+d] + \frac{1}{2(a-c)}\operatorname{sh}[(a-c)x+b-d].$$

**3.** $$\int \operatorname{sh}(ax+b)\operatorname{ch}(cx+d)\,dx = \frac{1}{2(a+c)}\operatorname{ch}[(a+c)x+b+d] + \frac{1}{2(a-c)}\operatorname{ch}[(a-c)x+b-d].$$

**4.** $$\int \operatorname{sh}(ax+b)\operatorname{sh}(ax+d)\,dx = -\frac{x}{2}\operatorname{ch}(b-d) + \frac{1}{4a}\operatorname{sh}(2ax+b+d).$$

**5.** $$\int \operatorname{ch}(ax+b)\operatorname{ch}(ax+d)\,dx = \frac{x}{2}\operatorname{ch}(b-d) + \frac{1}{4a}\operatorname{sh}(2ax+b+d).$$

**6.** $$\int \operatorname{sh}(ax+b)\operatorname{ch}(ax+d)\,dx = \frac{x}{2}\operatorname{sh}(b-d) + \frac{1}{4a}\operatorname{ch}(2ax+b+d).$$

**4.13. Интегралы вида** $\int \operatorname{sh}^p x \operatorname{sh} ax\,dx$.

**1.** $$\int \operatorname{sh}^p x \operatorname{sh} ax\,dx = \frac{1}{p+a}\left[\operatorname{sh}^p x \operatorname{ch} ax - p\int \operatorname{sh}^{p-1} x \operatorname{ch}(a-1)x\,dx\right].$$

**2.** $$\int \operatorname{sh}^p x \operatorname{sh} 2nx\,dx = 2n\left[\frac{\operatorname{sh}^{p+2} x}{p+2} + \sum_{k=1}^{n-1}\frac{2^{2k}(n^2-1^2)(n^2-2^2)\ldots(n^2-k^2)}{(2k+1)!(2k+p+2)}\operatorname{sh}^{2k+p+2} x\right]$$ $$[p \neq -2,\ -4,\ \ldots,\ -2n].$$

**3.** $$\int \operatorname{sh}^p x \operatorname{sh}(2n+1)x\,dx = (2n+1)\left[\int \operatorname{sh}^{p+1} x\,dx + \sum_{k=1}^{n}\frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\times \int \operatorname{sh}^{2k+p+1} x\,dx\right].$$

**4.** $$\int \operatorname{sh}^p x \operatorname{sh}(p+2)x\,dx = \frac{1}{p+1}\operatorname{sh}^{p+1} x \operatorname{sh}(p+1)x.$$

**5.** $$\int \frac{\operatorname{sh} 2nx}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1}\frac{\operatorname{sh}(2n-2k-1)x}{2n-2k-1}.$$

6. $\displaystyle\int \frac{\operatorname{sh}(2n+1)x}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1} \frac{\operatorname{sh}(2n-2k)x}{2n-2k} + x.$

7. $\displaystyle\int \frac{\operatorname{sh} 2x}{\operatorname{sh}^n x}\,dx = \frac{2}{2-n}\operatorname{sh}^{2-n} x \qquad [n \neq 2].$

8. $\displaystyle\int \frac{\operatorname{sh} 2x}{\operatorname{sh}^2 x}\,dx = 2\ln|\operatorname{sh} x|.$

9. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{sh} x}\,dx = \operatorname{sh} 2x + x.$

10. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{sh}^2 x}\,dx = 4\operatorname{ch} x + 3\ln\left|\operatorname{th}\frac{x}{2}\right|.$

11. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{sh}^3 x}\,dx = 4x - 3\operatorname{cth} x.$

### 4.14. Интегралы вида $\int \operatorname{sh}^p x \operatorname{ch} ax\,dx$.

1. $\displaystyle\int \operatorname{sh}^p x \operatorname{ch} ax\,dx = \frac{1}{p+a}\left[\operatorname{sh}^p x \operatorname{sh} ax - p\int \operatorname{sh}^{p-1} x \operatorname{sh}(a-1)x\,dx\right].$

2. $$\int \operatorname{sh}^p x \operatorname{ch} 2nx\,dx = \int \operatorname{sh}^p x\,dx + \sum_{k=1}^{n} \frac{2^{2k} n^2 (n^2-1^2)\ldots[n^2-(k-1)^2]}{(2k)!} \int \operatorname{sh}^{2k+p} x\,dx.$$

3. $$\int \operatorname{sh}^p x \operatorname{ch}(2n+1)x\,dx = \frac{1}{p+1}\operatorname{sh}^{p+1} x + \sum_{k=1}^{n} \frac{[(2n+1)^2-1][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \operatorname{sh}^{2k+p+1} x$$
$[p \neq -3,\ -5,\ \ldots,\ -2n-1].$

4. $\displaystyle\int \operatorname{sh}^p x \operatorname{ch}(p+2)x\,dx = \frac{1}{p+1}\operatorname{sh}^{p+1} x \operatorname{ch}(p+1)x.$

5. $\displaystyle\int \frac{\operatorname{ch} 2nx}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1} \frac{\operatorname{ch}(2n-2k-1)x}{2n-2k-1} + \ln\left|\operatorname{th}\frac{x}{2}\right|.$

6. $\displaystyle\int \frac{\operatorname{ch}(2n+1)x}{\operatorname{sh} x}\,dx = 2\sum_{k=0}^{n-1} \frac{\operatorname{ch}(2n-2k)x}{2n-2k} + \ln|\operatorname{sh} x|.$

7. $\displaystyle\int \frac{\operatorname{ch} 2x}{\operatorname{sh} x}\,dx = 2\operatorname{ch} x + \ln\left|\operatorname{th}\frac{x}{2}\right|.$

8. $\displaystyle\int \frac{\operatorname{ch} 2x}{\operatorname{sh}^2 x}\,dx = -\operatorname{cth} x + 2x.$

9. $\displaystyle\int \frac{\operatorname{ch} 2x}{\operatorname{sh}^3 x}\,dx = -\frac{\operatorname{ch} x}{2\operatorname{sh}^2 x} + \frac{3}{2}\ln\left|\operatorname{th}\frac{x}{2}\right|.$

10. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{sh}^n x}\,dx = \frac{4}{3-n}\operatorname{sh}^{3-n} x + \frac{1}{1-n}\operatorname{sh}^{1-n} x \qquad [n \neq 1,\ 3].$

11. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{sh} x}\,dx = 2\operatorname{sh}^2 x + \ln|\operatorname{sh} x|.$

12. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{sh}^3 x}\,dx = -\frac{1}{2}\operatorname{cth}^2 x + 4\ln|\operatorname{sh} x|.$

### 4.15. Интегралы вида $\int \operatorname{ch}^p x \operatorname{sh} ax\,dx$.

1. $\displaystyle\int \operatorname{ch}^p x \operatorname{sh} ax\,dx = \frac{1}{p+a}\left[\operatorname{ch}^p x \operatorname{ch} ax + p\int \operatorname{ch}^{p-1} x \operatorname{sh}(a-1)x\,dx\right].$

2. $\displaystyle\int \operatorname{ch}^p x \operatorname{sh} 2nx\,dx =$

$$= (-1)^{n+1}\left[\frac{\operatorname{ch}^{p+2} x}{p+2} + \sum_{k=1}^{n-1}(-4)^k \frac{(n^2-1^2)(n^2-2^2)\ldots(n^2-k^2)}{(2k+1)!\,(2k+p+2)}\operatorname{ch}^{2k+p+2} x\right]$$

$[p \neq -2,\ -4,\ \ldots,\ -2n].$

3. $\displaystyle\int \operatorname{ch}^p x \operatorname{sh}(2n+1)x\,dx = \frac{(-1)^n}{p+1}\operatorname{ch}^{p+1} x + \sum_{k=1}^{n}(-1)^{n-k} \times$

$$\times \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!\,(2k+p+1)}\operatorname{ch}^{2k+p+1} x$$

$[p \neq -3,\ -5,\ \ldots,\ -2n-1].$

4. $\displaystyle\int \operatorname{ch}^p x \operatorname{sh}(p+2)x\,dx = \frac{1}{p+1}\operatorname{ch}^{p+1} x \operatorname{ch}(p+1)x.$

5. $\displaystyle\int \frac{\operatorname{sh} 2nx}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\operatorname{ch}(2n-2k-1)x.$

6. $\displaystyle\int \frac{\operatorname{sh}(2n+1)x}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k}\operatorname{ch}(2n-2k)x + (-1)^n\ln\operatorname{ch} x.$

7. $\displaystyle\int \frac{\operatorname{sh} 2x}{\operatorname{ch}^n x}\,dx = \frac{2}{2-n}\operatorname{ch}^{2-n} x$ $[n \neq 2].$

8. $\displaystyle\int \frac{\operatorname{sh} 2x}{\operatorname{ch}^2 x}\,dx = 2\ln\operatorname{ch} x.$

9. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{ch}^n x}\,dx = \frac{4}{3-n}\operatorname{ch}^{3-n} x - \frac{1}{1-n}\operatorname{ch}^{1-n} x$ $[n \neq 1,\ 3].$

10. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{ch} x}\,dx = 2\operatorname{sh}^2 x - \ln\operatorname{ch} x.$

11. $\displaystyle\int \frac{\operatorname{sh} 3x}{\operatorname{ch}^3 x}\,dx = -\frac{1}{2}\operatorname{th}^2 x + 4\ln\operatorname{ch} x.$

### 4.16. Интегралы вида $\int \operatorname{ch}^p x \operatorname{ch} ax\,dx$.

1. $\displaystyle\int \operatorname{ch}^p x \operatorname{ch} ax\,dx = \frac{1}{p+a}\left[\operatorname{ch}^p x \operatorname{sh} ax + p\int \operatorname{ch}^{p-1} x \operatorname{ch}(a-1)x\,dx\right].$

2. $\displaystyle\int \operatorname{ch}^p x \operatorname{ch} 2nx\,dx =$

$$= (-1)^n\int \operatorname{ch}^p x\,dx + \sum_{k=1}^{n}(-4)^k \frac{n^2(n^2-1^2)\ldots[n^2-(k-1)^2]}{(2k)!}\int \operatorname{ch}^{2k+p} x\,dx.$$

3. $\displaystyle\int \operatorname{ch}^p x \operatorname{ch}(2n+1)x\,dx = (-1)^n(2n+1)\left\{\int \operatorname{ch}^{p+1} x\,dx + \sum_{k=1}^{n}(-1)^k \times \right.$

$\displaystyle\left. \times \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\int \operatorname{ch}^{2k+p+1} x\,dx\right\}.$

4. $\displaystyle\int \operatorname{ch}^p x \operatorname{ch}(p+2)x\,dx = \frac{1}{p+1}\operatorname{ch}^{p+1} x \operatorname{sh}(p+1)x.$

5. $\displaystyle\int \frac{\operatorname{ch} 2nx}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k-1}\operatorname{sh}(2n-2k-1)x + (-1)^n \arcsin \operatorname{th} x.$

6. $\displaystyle\int \frac{\operatorname{ch}(2n+1)x}{\operatorname{ch} x}\,dx = 2\sum_{k=0}^{n-1}\frac{(-1)^k}{2n-2k}\operatorname{sh}(2n-2k)x + (-1)^n x.$

7. $\displaystyle\int \frac{\operatorname{ch} 2x}{\operatorname{ch} x}\,dx = 2\operatorname{sh} x - \arcsin \operatorname{th} x.$

8. $\displaystyle\int \frac{\operatorname{ch} 2x}{\operatorname{ch}^2 x}\,dx = -\operatorname{th} x + 2x.$

9. $\displaystyle\int \frac{\operatorname{ch} 2x}{\operatorname{ch}^3 x}\,dx = -\frac{\operatorname{sh} x}{2\operatorname{ch}^2 x} + \frac{3}{2}\arcsin \operatorname{th} x.$

10. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{ch} x}\,dx = \operatorname{sh} 2x - x.$

11. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{ch}^2 x}\,dx = 4\operatorname{sh} x - 3\arcsin \operatorname{th} x.$

12. $\displaystyle\int \frac{\operatorname{ch} 3x}{\operatorname{ch}^3 x}\,dx = 4x - 3\operatorname{th} x.$

### 4.17. Интегралы вида $\int \sqrt{\operatorname{th} x}\,dx$, $\int \sqrt{\operatorname{cth} x}\,dx$.

1. $\displaystyle\int \sqrt{\operatorname{th} x}\,dx = \operatorname{Arth}\sqrt{\operatorname{th} x} - \operatorname{arctg}\sqrt{\operatorname{th} x}.$

2. $\displaystyle\int \sqrt{\operatorname{cth} x}\,dx = \operatorname{Arcth}\sqrt{\operatorname{cth} x} - \operatorname{arctg}\sqrt{\operatorname{cth} x}.$

### 4.18. Интегралы вида $\int x^p \operatorname{sh}^q x\,dx$.

1. $\displaystyle\int x^p \operatorname{sh}^q x\,dx = \frac{x^p}{q}\operatorname{sh}^{q-1} x \operatorname{ch} x - \frac{px^{p-1}}{q^2}\operatorname{sh}^q x +$

$\displaystyle + \frac{p(p-1)}{q^2}\int x^{p-2}\operatorname{sh}^q x\,dx - \frac{q-1}{q}\int x^p \operatorname{sh}^{q-2} x\,dx.$

2. $\displaystyle\int x^m \operatorname{sh}^{2n} x\,dx = \frac{(-1)^n C_{2n}^n}{2^{2n}(m+1)}x^{m+1} +$

$\displaystyle + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k C_{2n}^k \int x^m \operatorname{ch}(2n-2k)x\,dx.$

3. $\displaystyle\int x^m \operatorname{sh}^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n}(-1)^k C_{2n+1}^k \int x^m \operatorname{sh}(2n-2k+1)x\,dx.$

**4.** $$\int x \operatorname{sh}^{2n} x\,dx = \frac{1}{2n+1}\sum_{k=0}^{n-1}(-1)^k \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2^{k+1}n(n-1)\ldots(n-k)} \times \\ \times\left(-\frac{\operatorname{sh}^{2n-2k} x}{2n-2k} + x\operatorname{sh}^{2n-2k-1} x \operatorname{ch} x\right) + (-1)^n\frac{(2n-1)!!}{(2n)!!}\,\frac{x^2}{2}.$$

**5.** $$\int x\operatorname{sh}^{2n+1} x\,dx = \frac{1}{n+1}\sum_{k=0}^{n}(-1)^k\frac{2^k(n+1)n\ldots(n-k+1)}{(2n+1)(2n-1)\ldots(2n-2k+1)} \times \\ \times\left(-\frac{\operatorname{sh}^{2n-2k+1} x}{2n-2k+1} + x\operatorname{sh}^{2n-2k} x\operatorname{ch} x\right).$$

**6.** $$\int x^p\operatorname{sh} x\,dx = x^p\operatorname{ch} x - p\int x^{p-1}\operatorname{ch} x\,dx.$$

**7.** $$\int x^{2n}\operatorname{sh} x\,dx = (2n)!\left[\sum_{k=0}^{n}\frac{x^{2k}}{(2k)!}\operatorname{ch} x - \sum_{k=0}^{n-1}\frac{x^{2k+1}}{(2k+1)!}\operatorname{sh} x\right].$$

**8.** $$\int x^{2n+1}\operatorname{sh} x\,dx = (2n+1)!\sum_{k=0}^{n}\left[\frac{x^{2k+1}}{(2k+1)!}\operatorname{ch} x - \frac{x^{2k}}{(2k)!}\operatorname{sh} x\right].$$

**9.** $$\int x\operatorname{sh} x\,dx = x\operatorname{ch} x - \operatorname{sh} x.$$

**10.** $$\int x^2\operatorname{sh} x\,dx = (x^2+2)\operatorname{ch} x - 2x\operatorname{sh} x.$$

**11.** $$\int x^n\operatorname{sh}^2 x\,dx = -\frac{x^{n+1}}{2(n+1)} + \\ + \frac{n!}{4}\left[\sum_{k=0}^{[n/2]}\frac{x^{n-2k}}{2^{2k}(n-2k)!}\operatorname{sh} 2x - \sum_{k=0}^{[(n-1)/2]}\frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!}\operatorname{ch} 2x\right].$$

**12.** $$\int x\operatorname{sh}^2 x\,dx = \frac{x}{4}\operatorname{sh} 2x - \frac{1}{8}\operatorname{ch} 2x - \frac{x^2}{4}.$$

**13.** $$\int x^2\operatorname{sh}^2 x\,dx = \frac{1}{4}\left(x^2+\frac{1}{2}\right)\operatorname{sh} 2x - \frac{x}{4}\operatorname{ch} 2x - \frac{x^3}{6}.$$

**14.** $$\int x^n\operatorname{sh}^3 x\,dx = \frac{n!}{4}\sum_{k=0}^{[n/2]}\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\operatorname{ch} 3x}{3^{2k+1}} - 3\operatorname{ch} x\right) - \\ - \frac{n!}{4}\sum_{k=0}^{[(n-1)/2]}\frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\operatorname{sh} 3x}{3^{2k+2}} - 3\operatorname{sh} x\right).$$

**15.** $$\int x\operatorname{sh}^3 x\,dx = \frac{3}{4}\operatorname{sh} x - \frac{1}{36}\operatorname{sh} 3x - \frac{3}{4}x\operatorname{ch} x - \frac{x}{12}\operatorname{ch} 3x.$$

**16.** $$\int x^2\operatorname{sh}^3 x\,dx = -\left(\frac{3x^2}{4}+\frac{3}{2}\right)\operatorname{ch} x + \left(\frac{x^2}{12}+\frac{1}{54}\right)\operatorname{ch} 3x + \\ + \frac{3}{2}x\operatorname{sh} x - \frac{x}{18}\operatorname{sh} 3x.$$

## 4.19. Интегралы вида $\int \frac{x^p}{\operatorname{sh}^q x}\,dx$.

**1.** $$\int \frac{x^p}{\operatorname{sh}^q x}\,dx = -\frac{px^{p-1}}{(q-1)(q-2)\operatorname{sh}^{q-2} x} - \frac{x^p \operatorname{ch} x}{(q-1)\operatorname{sh}^{q-1} x} + \frac{p(p-1)}{(q-1)(q-2)} \int \frac{x^{p-2}}{\operatorname{sh}^{q-2} x}\,dx - \frac{q-2}{q-1} \int \frac{x^p}{\operatorname{sh}^{q-2} x}\,dx.$$

**2.** $$\int \frac{x^p}{\operatorname{sh} x}\,dx = \sum_{k=0}^{\infty} \frac{(2-2^{2k})B_{2k}}{(2k)!(2k+p)}\, x^{2k+p} \qquad [|x| < \pi;\ p > 0].$$

**3.** $$\int \frac{x^p}{\operatorname{sh}^2 x}\,dx = -x^p \operatorname{cth} x + p \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!\,(2k+p-1)}\, x^{2k+p-1} \qquad [|x| < \pi;\ p > 1].$$

**4.** $$\int \frac{x}{\operatorname{sh}^{2n} x}\,dx = \frac{1}{2n} \sum_{k=1}^{n-1} (-1)^k \frac{2n(2n-2)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+1)} \times \left[\frac{x \operatorname{ch} x}{\operatorname{sh}^{2n-2k+1} x} + \frac{1}{(2n-2k)\operatorname{sh}^{2n-2k} x}\right] - (-1)^n \frac{(2n-2)!!}{(2n-1)!!} \int \frac{x}{\operatorname{sh}^2 x}\,dx \qquad [n > 1].$$

**5.** $$\int \frac{x}{\operatorname{sh}^{2n+1} x}\,dx = \frac{1}{2n+1} \sum_{k=1}^{n} (-1)^k \frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2n(2n-2)\ldots(2n-2k+2)} \times \left[\frac{x \operatorname{ch} x}{\operatorname{sh}^{2n-2k+2} x} + \frac{1}{(2n-2k+1)\operatorname{sh}^{2n-2k+1} x}\right] + (-1)^n \frac{(2n-1)!!}{(2n)!!} \int \frac{x}{\operatorname{sh} x}\,dx.$$

**6.** $$\int \frac{x}{\operatorname{sh} x}\,dx = \sum_{k=0}^{\infty} \frac{2-2^{2k}}{(2k)!(2k+1)}\, B_{2k} x^{2k+1} \qquad [|x| < \pi].$$

**7.** $$\int \frac{x}{\operatorname{sh}^2 x}\,dx = -x \operatorname{cth} x + \ln|\operatorname{sh} x|.$$

**8.** $$\int \frac{x}{\operatorname{sh}^3 x}\,dx = -\frac{x \operatorname{ch} x}{2\operatorname{sh}^2 x} - \frac{1}{2\operatorname{sh} x} - \frac{1}{2} \int \frac{x}{\operatorname{sh} x}\,dx.$$

**9.** $$\int \frac{x}{\operatorname{sh}^4 x}\,dx = -\frac{x \operatorname{ch} x}{3\operatorname{sh}^3 x} - \frac{1}{6\operatorname{sh}^2 x} + \frac{2x}{3} \operatorname{cth} x - \frac{2}{3} \ln|\operatorname{sh} x|.$$

## 4.20. Интегралы вида $\int \frac{dx}{x^p \operatorname{sh}^q x}$.

**1.** $$\int \frac{dx}{x^p \operatorname{sh}^q x} = \frac{p}{(q-1)(q-2)x^{p+1}\operatorname{sh}^{q-2} x} - \frac{\operatorname{ch} x}{(q-1)x^p \operatorname{sh}^{q-1} x} - \frac{q-2}{q-1} \int \frac{dx}{x^p \operatorname{sh}^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)} \int \frac{dx}{x^{p+2}\operatorname{sh}^{q-2} x}.$$

**2.** $$\int \frac{dx}{x^n \operatorname{sh} x} = -[1+(-1)^n]\frac{2^{n-1}-1}{n!}\, B_n \ln|x| + \sum_{\substack{k=0\\ k\neq n/2}}^{\infty} \frac{2-2^{2k}}{(2k)!(2k-n)}\, B_{2k} x^{2k-n} \qquad [|x| < \pi;\ n \geqslant 1].$$

3. $$\int \frac{dx}{x^n \operatorname{sh}^2 x} = -\frac{\operatorname{cth} x}{x^n} - [1-(-1)^n]\frac{2^n n}{(n+1)!} B_{n+1} \ln|x| - \\ - \frac{n}{x^{n+1}} \sum_{\substack{k=0 \\ k\neq(n+1)/2}}^{\infty} \frac{B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \qquad [|x|<\pi].$$

## 4.21. Интегралы вида $\int x^p \operatorname{ch}^q x\, dx$.

1. $$\int x^p \operatorname{ch}^q x\, dx = \frac{x^p}{q} \operatorname{sh} x \operatorname{ch}^{q-1} x - \frac{px^{p-1}}{q^2} \operatorname{ch}^q x + \\ + \frac{p(p-1)}{q^2} \int x^{p-2} \operatorname{ch}^q x\, dx + \frac{q-1}{q} \int x^p \operatorname{ch}^{q-2} x\, dx.$$

2. $$\int x^m \operatorname{ch}^{2n} x\, dx = \frac{C_{2n}^n}{2^{2n}(m+1)} x^{m+1} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} C_{2n}^k \int x^m \operatorname{ch}(2n-2k)x\, dx.$$

3. $$\int x^m \operatorname{ch}^{2n+1} x\, dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} C_{2n+1}^k \int x^m \operatorname{ch}(2n-2k+1)x\, dx.$$

4. $$\int x \operatorname{ch}^{2n} x\, dx = \frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2^{k+1} n(n-1)\ldots(n-k)} \times \\ \times \left(-\frac{\operatorname{ch}^{2n-2k} x}{2n-2k} + x \operatorname{sh} x \operatorname{ch}^{2n-2k-1} x\right) + \frac{(2n-1)!!}{(2n)!!} \frac{x^2}{2}.$$

5. $$\int x \operatorname{ch}^{2n+1} x\, dx = \frac{1}{n+1} \sum_{k=0}^{n} \frac{2^k (n+1)n\ldots(n-k+1)}{(2n+1)(2n-1)\ldots(2n-2k+1)} \times \\ \times \left(-\frac{\operatorname{ch}^{2n-2k+1} x}{2n-2k+1} + x \operatorname{sh} x \operatorname{ch}^{2n-2k} x\right).$$

6. $$\int x^p \operatorname{ch} x\, dx = x^p \operatorname{sh} x - p \int x^{p-1} \operatorname{sh} x\, dx.$$

7. $$\int x^{2n} \operatorname{ch} x\, dx = (2n)! \left[\sum_{k=0}^{n} \frac{x^{2k}}{(2k)!} \operatorname{sh} x - \sum_{k=0}^{n-1} \frac{x^{2k+1}}{(2k+1)!} \operatorname{ch} x\right].$$

8. $$\int x^{2n+1} \operatorname{ch} x\, dx = (2n+1)! \sum_{k=0}^{n} \left[\frac{x^{2k+1}}{(2k+1)!} \operatorname{sh} x - \frac{x^{2k}}{(2k)!} \operatorname{ch} x\right].$$

9. $$\int x \operatorname{ch} x\, dx = x \operatorname{sh} x - \operatorname{ch} x.$$

10. $$\int x^2 \operatorname{ch} x\, dx = (x^2+2) \operatorname{sh} x - 2x \operatorname{ch} x.$$

11. $$\int x^n \operatorname{ch}^2 x\, dx = \frac{x^{n+1}}{2(n+1)} + \\ + \frac{n!}{4} \left[\sum_{k=0}^{[n/2]} \frac{x^{n-2k}}{2^{2k}(n-2k)!} \operatorname{sh} 2x - \sum_{k=0}^{[(n-1)/2]} \frac{x^{n-2k-1}}{2^{2k+1}(n-2k-1)!} \operatorname{ch} 2x\right].$$

12. $$\int x \operatorname{ch}^2 x\, dx = \frac{x}{4} \operatorname{sh} 2x - \frac{1}{8} \operatorname{ch} 2x + \frac{x^2}{4}.$$

**13.** $\displaystyle\int x^2 \operatorname{ch}^2 x\,dx = \frac{1}{4}\left(x^2+\frac{1}{2}\right)\operatorname{sh}2x - \frac{x}{4}\operatorname{ch}2x + \frac{x^3}{6}.$

**14.** $$\int x^n \operatorname{ch}^3 x\,dx = \frac{n!}{4}\sum_{k=0}^{[n/2]}\frac{x^{n-2k}}{(n-2k)!}\left(\frac{\operatorname{sh}3x}{3^{2k+1}} - 3\operatorname{sh}x\right) - \\ - \frac{n!}{4}\sum_{k=0}^{[(n-1)/2]}\frac{x^{n-2k-1}}{(n-2k-1)!}\left(\frac{\operatorname{ch}3x}{3^{2k+2}} - 3\operatorname{ch}x\right).$$

**15.** $\displaystyle\int x\operatorname{ch}^3 x\,dx = -\frac{3}{4}\operatorname{ch}x - \frac{1}{36}\operatorname{ch}3x + \frac{3}{4}x\operatorname{sh}x + \frac{x}{12}\operatorname{sh}3x.$

**16.** $$\int x^2\operatorname{ch}^3 x\,dx = \left(\frac{3x^2}{4}+\frac{3}{2}\right)\operatorname{sh}x + \left(\frac{x^2}{12}+\frac{1}{54}\right)\operatorname{sh}3x - \\ - \frac{3}{2}x\operatorname{ch}x - \frac{x}{18}\operatorname{ch}3x.$$

### 4.22. Интегралы вида $\displaystyle\int \frac{x^p}{\operatorname{ch}^q x}\,dx$.

**1.** $$\int\frac{x^p}{\operatorname{ch}^q x}\,dx = \frac{px^{p-1}}{(q-1)(q-2)\operatorname{ch}^{q-2}x} + \frac{x^p\operatorname{sh}x}{(q-1)\operatorname{ch}^{q-1}x} - \\ - \frac{p(p-1)}{(q-1)(q-2)}\int\frac{x^{p-2}}{\operatorname{ch}^{q-2}x}\,dx + \frac{q-2}{q-1}\int\frac{x^p}{\operatorname{ch}^{q-2}x}\,dx.$$

**2.** $\displaystyle\int\frac{x^p}{\operatorname{ch}x}\,dx = \sum_{k=0}^{\infty}\frac{E_{2k}x^{2k+p+1}}{(2k)!(2k+p+1)}$ $\quad[|x|<\pi/2;\ p\geqslant].$

**3.** $\displaystyle\int\frac{x^p}{\operatorname{ch}^2 x}\,dx = x^p\operatorname{th}x - p\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!\,(2k+p-1)}\,x^{2k+p-1}$

$[|x|<\pi/2;\ p\geqslant 1].$

**4.** $$\int\frac{x}{\operatorname{ch}^{2n}x}\,dx = \frac{1}{2n}\sum_{k=1}^{n-1}\frac{2n(2n-2)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+1)}\times \\ \times\left[\frac{x\operatorname{sh}x}{\operatorname{ch}^{2n-2k+1}x} + \frac{1}{(2n-2k)\operatorname{ch}^{2n-2k}x}\right] + \frac{(2n-2)!!}{(2n-1)!!}\int\frac{x}{\operatorname{ch}^2 x}\,dx \qquad [n>1].$$

**5.** $$\int\frac{x}{\operatorname{ch}^{2n+1}x}\,dx = \frac{1}{2n+1}\sum_{k=1}^{n}\frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2n(2n-2)\ldots(2n-2k+2)}\times \\ \times\left[\frac{x\operatorname{sh}x}{\operatorname{ch}^{2n-2k+2}x} + \frac{1}{(2n-2k+1)\operatorname{ch}^{2n-2k+1}x}\right] + \frac{(2n-1)!!}{(2n)!!}\int\frac{x}{\operatorname{ch}x}\,dx.$$

**6.** $\displaystyle\int\frac{x}{\operatorname{ch}x}\,dx = \sum_{k=0}^{\infty}\frac{E_{2k}x^{2k+2}}{(2k)!(2k+2)}$ $\quad[|x|<\pi/2].$

**7.** $\displaystyle\int\frac{x}{\operatorname{ch}^2 x}\,dx = x\operatorname{th}x - \ln\operatorname{ch}x.$

**8.** $\displaystyle\int\frac{x}{\operatorname{ch}^3 x}\,dx = \frac{x\operatorname{sh}x}{2\operatorname{ch}^2 x} + \frac{1}{2\operatorname{ch}x} + \frac{1}{2}\int\frac{x}{\operatorname{ch}x}\,dx.$

**9.** $\displaystyle\int\frac{x}{\operatorname{ch}^4 x}\,dx = \frac{x\operatorname{sh}x}{3\operatorname{ch}^3 x} + \frac{1}{6\operatorname{ch}^2 x} + \frac{2x}{3}\operatorname{th}x - \frac{2}{3}\ln\operatorname{ch}x.$

**4.23. Интегралы вида** $\int \frac{dx}{x^p \operatorname{ch}^q x}$.

**1.** $$\int \frac{dx}{x^p \operatorname{ch}^q x} = \frac{-p}{(q-1)(q-2)x^{p+1} \operatorname{ch}^{q-2} x} + \frac{\operatorname{sh} x}{(q-1)x^p \operatorname{ch}^{q-1} x} + \frac{q-2}{q-1}\int \frac{dx}{x^p \operatorname{ch}^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)}\int \frac{dx}{x^{p+2} \operatorname{ch}^{q-2} x}.$$

**2.** $$\int \frac{dx}{x^n \operatorname{ch} x} = \sum_{\substack{k=0 \\ k\neq(n-1)/2}}^{\infty} \frac{E_{2k}}{(2k)!(2k-n+1)} x^{2k-n+1} + \frac{1}{2}\left[1-(-1)^n\right]\frac{E_{n-1}}{(n-1)!}\ln|x| \qquad [|x|<\pi/2].$$

**3.** $$\int \frac{dx}{x^n \operatorname{ch}^2 x} = \frac{\operatorname{th} x}{x^n} + \left[1-(-1)^n\right]\frac{2^n(2^{n+1}-1)n}{(n+1)!} B_{n+1}\ln|x| + \frac{n}{x^{n+1}} \sum_{\substack{k=1 \\ k\neq(n+1)/2}}^{\infty} \frac{(2^{2k}-1)B_{2k}}{(2k)!(2k-n-1)} (2x)^{2k} \qquad [|x|<\pi/2].$$

**4.24. Интегралы вида** $\int x^r \operatorname{sh}^p x \operatorname{ch}^q x\, dx$.

**1.** $$\int x^r \operatorname{sh}^p x \operatorname{ch}^q x\, dx = \frac{1}{(p+q)^2}\Big[(p+q)x^r \operatorname{sh}^{p+1} x \operatorname{ch}^{q-1} x - rx^{r-1}\operatorname{sh}^p x \operatorname{ch}^q x + r(r+1)\int x^{r-2}\operatorname{sh}^p x \operatorname{ch}^q x\, dx + rp\int x^{r-1}\operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x\, dx + (q-1)(p+q)\int x^r \operatorname{sh}^p x \operatorname{ch}^{q-2} x\, dx\Big].$$

**2.** $$= \frac{1}{(p+q)^2}\Big[(p+q)x^r \operatorname{sh}^{p-1} x \operatorname{ch}^{q+1} x - rx^{r-1}\operatorname{sh}^p x \operatorname{ch}^q x + r(r-1)\int x^{r-2}\operatorname{sh}^p x \operatorname{ch}^q x\, dx - rq\int x^{r-1}\operatorname{sh}^{p-1} x \operatorname{ch}^{q-1} x\, dx - (p-1)(p+q)\int x^r \operatorname{sh}^{p-2} x \operatorname{ch}^q x\, dx\Big].$$

**3.** $$\int x^p \frac{\operatorname{sh}^{2m} x}{\operatorname{ch}^n x}\, dx = \sum_{k=0}^{m}(-1)^{m+k} C_m^k \int \frac{x^p}{\operatorname{ch}^{n-2k} x}\, dx.$$

**4.** $$\int x^p \frac{\operatorname{sh}^{2m+1} x}{\operatorname{ch}^n x}\, dx = \sum_{k=0}^{m}(-1)^{m+k} C_m^k \int \frac{x^p \operatorname{sh} x}{\operatorname{ch}^{n-2k} x}\, dx.$$

**5.** $$\int x^p \frac{\operatorname{sh} x}{\operatorname{ch}^n x}\, dx = -\frac{x^p}{(n-1)\operatorname{ch}^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}}{\operatorname{ch}^{n-1} x}\, dx.$$

**6.** $$\int x \frac{\operatorname{sh} x}{\operatorname{ch}^2 x}\, dx = -\frac{x}{\operatorname{ch} x} + \operatorname{arctg}\operatorname{sh} x.$$

**7.** $$\int x^p \frac{\operatorname{ch}^{2m} x}{\operatorname{sh}^n x}\, dx = \sum_{k=0}^{m} C_m^k \int \frac{x^p}{\operatorname{sh}^{n-2k} x}\, dx.$$

**8.** $$\int x^p \frac{\operatorname{ch}^{2m+1} x}{\operatorname{sh}^n x}\, dx = \sum_{k=0}^{m} C_m^k \int \frac{x^p \operatorname{ch} x}{\operatorname{sh}^{n-2k} x}\, dx.$$

9. $\displaystyle\int x^p \frac{\operatorname{ch} x}{\operatorname{sh}^n x}\,dx = -\frac{x^p}{(n-1)\operatorname{sh}^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}}{\operatorname{sh}^{n-1} x}\,dx.$

10. $\displaystyle\int x \frac{\operatorname{ch} x}{\operatorname{sh}^2 x}\,dx = -\frac{x}{\operatorname{sh} x} + \ln\left|\operatorname{th}\frac{x}{2}\right|.$

**4.25.** Интегралы вида $\displaystyle\int x^p \operatorname{th}^n x\,dx.$

1. $\displaystyle\int x^p \operatorname{th} x\,dx = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)B_{2k}}{(2k)!\,(2k+p)}\,x^{2k+p}$ $[|x| < \pi/2;\ p \geqslant -1].$

2. $\displaystyle\int x^p \operatorname{th}^{2n} x\,dx = \frac{x^{p+1}}{p+1} + \sum_{k=0}^{n-1}(-1)^{n+k} C_n^k \int x^p \operatorname{ch}^{2k-2n} x\,dx.$

3. $$\int x^p \operatorname{th}^{2n+1} x\,dx = \int x^p \operatorname{th} x\,dx + \\ + \sum_{k=0}^{n-1} \frac{(-1)^{n+k}}{2n-2k} C_n^k \left(-x^p \operatorname{ch}^{2k-2n} x + p\int x^{p-1} \operatorname{ch}^{2k-2n} x\,dx\right).$$

**4.26.** Интегралы вида $\displaystyle\int x^p \operatorname{cth}^n x\,dx.$

1. $\displaystyle\int x^p \operatorname{cth} x\,dx = \sum_{k=0}^{\infty} \frac{2^{2k} B_{2k}}{(2k)!\,(2k+p)}\,x^{2k+p}$ $[|x| < \pi;\ p \geqslant 1].$

2. $\displaystyle\int x^p \operatorname{cth}^{2n} x\,dx = \frac{x^{p+1}}{p+1} + \sum_{k=0}^{n-1} C_n^k \int x^p \operatorname{sh}^{2k-2n} x\,dx.$

3. $$\int x^p \operatorname{cth}^{2n+1} x\,dx = \int x^p \operatorname{cth} x\,dx + \\ + \sum_{k=0}^{n-1} \frac{C_n^k}{2n-2k} \left(-x^p \operatorname{sh}^{2k-2n} x + p\int x^{p-1} \operatorname{sh}^{2k-2n} x\,dx\right).$$

**4.27.** Интегралы вида $\displaystyle\int \frac{x^p \operatorname{ch}^m x\,dx}{(a+b\operatorname{sh} x)^q},\ \int \frac{x^p \operatorname{sh}^n x\,dx}{(a+b\operatorname{ch} x)^q}.$

1. $\displaystyle\int \frac{x^p \operatorname{ch} x\,dx}{(a+b\operatorname{sh} x)^q} = -\frac{x^p}{(q-1)b(a+b\operatorname{sh} x)^{q-1}} + \frac{p}{(q-1)b}\int \frac{x^{p-1}\,dx}{(a+b\operatorname{sh} x)^{q-1}}.$

2. $\displaystyle\int \frac{x^p \operatorname{sh} x\,dx}{(a+b\operatorname{ch} x)^q} = -\frac{x^p}{(q-1)b(a+b\operatorname{ch} x)^{q-1}} + \frac{p}{(q-1)b}\int \frac{x^{p-1}\,dx}{(a+b\operatorname{ch} x)^{q-1}}.$

3. $\displaystyle\int \frac{x\,dx}{1+\operatorname{ch} x} = x\operatorname{th}\frac{x}{2} - 2\ln\operatorname{ch}\frac{x}{2}.$

4. $\displaystyle\int \frac{x\,dx}{1-\operatorname{ch} x} = x\operatorname{cth}\frac{x}{2} - 2\ln\left|\operatorname{sh}\frac{x}{2}\right|.$

5. $\displaystyle\int \frac{x\operatorname{sh} x\,dx}{(1+\operatorname{ch} x)^2} = -\frac{x}{\operatorname{ch} x+1} + \operatorname{th}\frac{x}{2}.$

6. $\displaystyle\int \frac{x\operatorname{sh} x\,dx}{(1-\operatorname{ch} x)^2} = -\frac{x}{\operatorname{ch} x-1} - \operatorname{cth}\frac{x}{2}.$

## 4.28. Интегралы вида $\int (bx+c)^{\pm n} \operatorname{sh} ax\, dx$.

**1.** $$\int (bx+c)^n \operatorname{sh} ax\, dx = \frac{b^n}{a^{n+1}} \left[ \operatorname{ch}\frac{ac}{b} \int t^n \operatorname{sh} t\, dt - \operatorname{sh}\frac{ac}{b} \int t^n \operatorname{ch} t\, dt \right]$$
$$[t = ax + ac/b].$$

**2.** $$\int (bx+c) \operatorname{sh} ax\, dx = \frac{1}{a}(bx+c)\operatorname{ch} ax - \frac{b}{a^2}\operatorname{sh} ax.$$

**3.** $$\int (bx+c)^2 \operatorname{sh} ax\, dx = \frac{1}{a^3}[a^2(bx+c)^2 + 2b^2]\operatorname{ch} ax - \frac{2b(bx+c)}{a^2}\operatorname{sh} ax.$$

**4.** $$\int (bx+c)^3 \operatorname{sh} ax\, dx = \\ = \frac{bx+c}{a^3}[a^2(bx+c)^2 + 6b^2]\operatorname{ch} ax - \frac{3b}{a^4}[a^2(bx+c)^2 + 2b^2]\operatorname{sh} ax.$$

**5.** $$\int \frac{\operatorname{sh} ax}{(bx+c)^n}\, dx = \frac{a^{n-1}}{b^n}\left( \operatorname{ch}\frac{ac}{b} \int \frac{\operatorname{sh} t}{t^n}\, dt - \operatorname{sh}\frac{ac}{b} \int \frac{\operatorname{ch} t}{t^n}\, dt \right)$$
$$[t = ax + ac/b].$$

## 4.29. Интегралы вида $\int (bx+c)^{\pm n} \operatorname{ch} ax\, dx$.

**1.** $$\int (bx+c)^n \operatorname{ch} ax\, dx = \frac{b^n}{a^{n+1}} \left[ \operatorname{ch}\frac{ac}{b} \int t^n \operatorname{ch} t\, dt - \operatorname{sh}\frac{ac}{b} \int t^n \operatorname{sh} t\, dt \right]$$
$$[t = ax + ac/b].$$

**2.** $$\int (bx+c) \operatorname{ch} ax\, dx = \frac{1}{a}(bx+c)\operatorname{sh} ax - \frac{b}{a^2}\operatorname{ch} ax.$$

**3.** $$\int (bx+c)^2 \operatorname{ch} ax\, dx = \frac{1}{a^3}[a^2(bx+c)^2 + 2b^2]\operatorname{sh} ax - \frac{2b(bx+c)}{a^2}\operatorname{ch} ax.$$

**4.** $$\int (bx+c)^3 \operatorname{ch} ax\, dx = \\ = \frac{bx+c}{a^3}[a^2(bx+c)^2 + 6b^2]\operatorname{sh} ax - \frac{3b}{a^4}[a^2(bx+c)^2 + 2b^2]\operatorname{ch} ax.$$

**5.** $$\int \frac{\operatorname{ch} ax}{(bx+c)^n}\, dx = \frac{a^{n-1}}{b^n}\left( \operatorname{ch}\frac{ac}{b} \int \frac{\operatorname{ch} t}{t^n}\, dt - \operatorname{sh}\frac{ac}{b} \int \frac{\operatorname{sh} t}{t^n}\, dt \right)$$
$$[t = ax + ac/b].$$

## 4.30. Интегралы вида $\int x^p e^{bx} \operatorname{sh} ax\, dx$.

**1.** $$\int x^p e^{bx} \operatorname{sh} ax\, dx = \frac{1}{2}\int x^p e^{(a+b)x}\, dx - \frac{1}{2}\int x^p e^{(b-a)x}\, dx.$$

**2.** $$\int e^{bx} \operatorname{sh} ax\, dx = \frac{e^{bx}}{a^2-b^2}(a \operatorname{ch} bx - b \operatorname{sh} ax).$$

**3.** $$\int x e^{bx} \operatorname{sh} ax\, dx = \frac{e^{bx}}{a^2-b^2}\left[ \left( ax + \frac{2ab}{a^2-b^2} \right)\operatorname{ch} ax - \left( bx + \frac{a^2+b^2}{a^2-b^2} \right)\operatorname{sh} ax \right].$$

**4.** $\displaystyle\int x^2 e^{bx}\,\mathrm{sh}\,ax\,dx = \frac{e^{bx}}{a^2-b^2}\left[ax^2 + \frac{4ab}{a^2-b^2}x + \frac{2a(a^2+3b^2)}{(a^2-b^2)^2}\right]\mathrm{ch}\,ax - {}$
$\displaystyle - \frac{e^{bx}}{a^2-b^2}\left[bx^2 + \frac{2(a^2+b^2)}{a^2-b^2}x + \frac{2b(3a^2+b^2)}{(a^2-b^2)^2}\right]\mathrm{sh}\,ax.$

**5.** $\displaystyle\int e^{ax}\,\mathrm{sh}\,ax\,dx = \frac{1}{4a}e^{2ax} - \frac{x}{2}.$

**6.** $\displaystyle\int xe^{ax}\,\mathrm{sh}\,ax\,dx = \frac{e^{2ax}}{4a}\left(x - \frac{1}{2a}\right) - \frac{x^2}{4}.$

**7.** $\displaystyle\int x^2 e^{ax}\,\mathrm{sh}\,ax\,dx = \frac{e^{2ax}}{4a}\left(x^2 + \frac{x}{a} + \frac{1}{2a^2}\right) - \frac{x^3}{6}.$

**8.** $\displaystyle\int e^{-ax}\,\mathrm{sh}\,ax\,dx = \frac{1}{4a}e^{-2ax} + \frac{x}{2}.$

**9.** $\displaystyle\int xe^{-ax}\,\mathrm{sh}\,ax\,dx = \frac{x^2}{4} + \frac{e^{-2ax}}{4a}\left(x + \frac{1}{2a}\right).$

**10.** $\displaystyle\int x^2 e^{-ax}\,\mathrm{sh}\,ax\,dx = \frac{x^3}{6} + \frac{e^{-2ax}}{4a}\left(x^2 + \frac{x}{a} + \frac{1}{2a^2}\right).$

### 4.31. Интегралы вида $\int x^p e^{bx}\,\mathrm{ch}\,ax\,dx$.

**1.** $\displaystyle\int x^p e^{bx}\,\mathrm{ch}\,ax\,dx = \frac{1}{2}\int x^p e^{(a+b)x}\,dx + \frac{1}{2}\int x^p e^{(b-a)x}\,dx.$

**2.** $\displaystyle\int e^{bx}\,\mathrm{ch}\,ax\,dx = \frac{e^{bx}}{a^2-b^2}(a\,\mathrm{sh}\,ax - b\,\mathrm{ch}\,ax).$

**3.** $\displaystyle\int xe^{bx}\,\mathrm{ch}\,ax\,dx = \frac{e^{bx}}{a^2-b^2}\left[\left(ax + \frac{2ab}{a^2-b^2}\right)\mathrm{sh}\,ax - \left(bx + \frac{a^2+b^2}{a^2-b^2}\right)\mathrm{ch}\,ax\right].$

**4.** $\displaystyle\int x^2 e^{bx}\,\mathrm{ch}\,ax\,dx = \frac{e^{bx}}{a^2-b^2}\left[ax^2 + \frac{4ab}{a^2-b^2}x + \frac{2a(a^2+3b^2)}{(a^2-b^2)^2}\right]\mathrm{sh}\,ax - {}$
$\displaystyle - \frac{e^{bx}}{a^2-b^2}\left[bx^2 + \frac{2(a^2+b^2)}{a^2-b^2}x + \frac{2b(3a^2+b^2)}{(a^2-b^2)^2}\right]\mathrm{ch}\,ax.$

**5.** $\displaystyle\int e^{ax}\,\mathrm{ch}\,ax\,dx = \frac{1}{4a}e^{2ax} + \frac{x}{2}.$

**6.** $\displaystyle\int xe^{ax}\,\mathrm{ch}\,ax\,dx = \frac{e^{2ax}}{4a}\left(x - \frac{1}{2a}\right) + \frac{x^2}{4}.$

**7.** $\displaystyle\int x^2 e^{ax}\,\mathrm{ch}\,ax\,dx = \frac{e^{2ax}}{4a}\left(x^2 - \frac{x}{a} + \frac{1}{2a^2}\right) + \frac{x^3}{6}.$

**8.** $\displaystyle\int e^{-ax}\,\mathrm{ch}\,ax\,dx = \frac{x}{2} - \frac{1}{4a}e^{-2ax}.$

**9.** $\displaystyle\int xe^{-ax}\,\mathrm{ch}\,ax\,dx = \frac{x^2}{4} - \frac{e^{-2ax}}{4a}\left(x + \frac{1}{2a}\right).$

**10.** $\displaystyle\int x^2 e^{-ax}\,\mathrm{ch}\,ax\,dx = \frac{x^3}{6} + \frac{e^{-2ax}}{4a}\left(x^2 + \frac{x}{a} + \frac{1}{2a^2}\right).$

# 5. ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

## 5.1. Введение.

Если $R$ — рациональная функция своих аргументов, то интегралы вида $\int R(\sin x,\, \cos x,\, \operatorname{tg} x,\, \operatorname{ctg} x)\, dx$ приводятся к интегралам от рациональных функций.

**1.** $$\int R(\sin x,\, \cos x,\, \operatorname{tg} x,\, \operatorname{ctg} x)\, dx = \\ = \int R\left(\frac{2t}{1+t^2},\, \frac{1-t^2}{1+t^2},\, \frac{2t}{1-t^2}\, \frac{1-t^2}{2t}\right) \frac{2\, dt}{1+t^2} \qquad \left[t = \operatorname{tg}\frac{x}{2}\right].$$

Если
$$R(\sin x,\, \cos x) = -R(-\sin x,\, \cos x),$$
то

**2.** $$\int R(\sin x,\, \cos x)\, dx = -\int R(\sqrt{1-t^2}\,,\, t) \frac{dt}{\sqrt{1-t^2}} \qquad [t = \cos x].$$

Если
$$R(\sin x,\, \cos x) = -R(\sin x,\, -\cos x),$$
то

**3.** $$\int R(\sin x,\, \cos x)\, dx = \int R(t,\, \sqrt{1-t^2}\,) \frac{dt}{\sqrt{1-t^2}} \qquad [t = \sin x].$$

Если
$$R(\sin x,\, \cos x) = R(-\sin x,\, -\cos x),$$
то

**4.** $$\int R(\sin x,\, \cos x)\, dx = \int R\left(\frac{t}{\sqrt{1+t^2}},\, \frac{1}{\sqrt{1+t^2}}\right) \frac{dt}{\sqrt{1+t^2}} \qquad [t = \operatorname{tg} x].$$

## 5.2. Интегралы вида $\int \sin^p x\, dx$.

**1.** $$\int \sin^p x\, dx = -\frac{1}{p}\sin^{p-1} x \cos x + \frac{p-1}{p}\int \sin^{p-2} x\, dx.$$

**2.** $$\int \sin^{2n} x\, dx = -\frac{\cos x}{2n}\left[\sin^{2n-1} x + \right. \\ \left. + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)} \sin^{2n-2k-1} x\right] + \frac{(2n-1)!!}{2^n n!}\, x.$$

**3.** $$= \frac{1}{2^{2n}} C_{2n}^n x + \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k C_{2n}^k \frac{\sin\,(2n-2k)x}{2n-2k}.$$

4. $$\int \sin^{2n+1} x\,dx = -\frac{\cos x}{2n+1}\left[\sin^{2n} x + \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1)\ldots(n-k)}{(2n-1)(2n-3)\ldots(2n-2k-1)} \sin^{2n-2k-2} x\right].$$

5. $$= \frac{(-1)^{n+1}}{2^{2n}} \sum_{k=0}^{n} (-1)^k C_{2n+1}^k \frac{\cos(2n-2k+1)x}{2n-2k+1}.$$

6. $$= \sum_{k=0}^{n} \frac{(-1)^{k+1}}{2k+1} C_n^k \cos^{2k+1} x.$$

7. $$\int \sin x\,dx = -\cos x.$$

8. $$\int \sin^2 x\,dx = -\frac{1}{4}\sin 2x + \frac{1}{2}x = -\frac{1}{2}\sin x\cos x + \frac{1}{2}x.$$

9. $$\int \sin^3 x\,dx = \frac{1}{12}\cos 3x - \frac{3}{4}\cos x = \frac{1}{3}\cos^3 x - \cos x.$$

10. $$\int \sin^4 x\,dx = \frac{3x}{8} - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x = -\frac{3}{8}\sin x\cos x - \frac{1}{4}\sin^3 x\cos x + \frac{3}{8}x.$$

11. $$\int \sin^5 x\,dx = -\frac{5}{8}\cos x + \frac{5}{48}\cos 3x - \frac{1}{80}\cos 5x = -\frac{1}{5}\sin^4 x\cos x + \frac{4}{15}\cos^3 x - \frac{4}{5}\cos x.$$

12. $$\int \sin^6 x\,dx = \frac{5}{16}x - \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x - \frac{1}{192}\sin 6x = -\frac{1}{6}\sin^5 x\cos x - \frac{5}{24}\sin^3 x\cos x - \frac{5}{16}\sin x\cos x + \frac{5}{16}x.$$

### 5.3. Интегралы вида $\int \frac{dx}{\sin^p x}$.

1. $$\int \frac{dx}{\sin^p x} = -\frac{\cos x}{(p-1)\sin^{p-1} x} + \frac{p-2}{p-1}\int \frac{dx}{\sin^{p-2} x}.$$

2. $$\int \frac{dx}{\sin^{2n} x} = -\frac{\cos x}{2n-1}\left[\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x\right].$$

3. $$\int \frac{dx}{\sin^{2n+1} x} = -\frac{\cos x}{2n}\left[\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)} \operatorname{cosec}^{2n-2k} x\right] + \frac{(2n-1)!!}{2^n n!}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

4. $$\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg}\frac{x}{2}\right| = \ln\left|\frac{1-\cos x}{\sin x}\right| = \ln\left|\frac{\sin x}{1+\cos x}\right| = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|.$$

5. $$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x.$$

6. $\displaystyle\int \frac{dx}{\sin^3 x} = -\frac{\cos x}{2\sin^2 x} + \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

7. $\displaystyle\int \frac{dx}{\sin^4 x} = -\frac{\cos x}{3\sin^3 x} - \frac{2}{3}\operatorname{ctg} x = -\frac{1}{3}\operatorname{ctg}^3 x - \operatorname{ctg} x.$

8. $\displaystyle\int \frac{dx}{\sin^5 x} = -\frac{\cos x}{4\sin^4 x} - \frac{3}{8}\,\frac{\cos x}{\sin^2 x} + \frac{3}{8}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

9. $\displaystyle\int \frac{dx}{\sin^6 x} = -\frac{\cos x}{5\sin^5 x} - \frac{4}{15}\operatorname{ctg}^3 x - \frac{4}{5}\operatorname{ctg} x = -\frac{1}{5}\operatorname{ctg}^5 x - \frac{2}{3}\operatorname{ctg}^3 x - \operatorname{ctg} x.$

**5.4. Интегралы вида** $\displaystyle\int \cos^p x\,dx$.

1. $\displaystyle\int \cos^p x\,dx = \frac{1}{p}\sin x\cos^{p-1} x + \frac{p-1}{p}\int \cos^{p-2} x\,dx.$

2. $$\int \cos^{2n} x\,dx = \frac{\sin x}{2n}\Bigg[\cos^{2n-1} x + \sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\dots(2n-2k+1)}{2^k(n-1)(n-2)\dots(n-k)}\cos^{2n-2k-1} x\Bigg] + \frac{(2n-1)!!}{2^n n!}\,x.$$

3. $$= \frac{1}{2^{2n}}C_{2n}^{n}x + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1}C_{2n}^{k}\frac{\sin(2n-2k)x}{2n-2k}.$$

4. $$\int \cos^{2n+1} x\,dx = \frac{\sin x}{2n+1}\Bigg[\cos^{2n} x + \sum_{k=0}^{n-1}\frac{2^{k+1}n(n-1)\dots(n-k)}{(2n-1)(2n-3)\dots(2n-2k-1)}\cos^{2n-2k-2} x\Bigg].$$

5. $$= \frac{1}{2^{2n}}\sum_{k=0}^{n}C_{2n+1}^{k}\frac{\sin(2n-2k+1)x}{2n-2k+1}.$$

6. $$= \sum_{k=0}^{n}\frac{(-1)^k}{2k+1}C_n^k\sin^{2k+1} x.$$

7. $\displaystyle\int \cos x\,dx = \sin x.$

8. $\displaystyle\int \cos^2 x\,dx = \frac{1}{4}\sin 2x + \frac{x}{2} = \frac{1}{2}\sin x\cos x + \frac{1}{2}x.$

9. $\displaystyle\int \cos^3 x\,dx = \frac{1}{12}\sin 3x + \frac{3}{4}\sin x = \sin x - \frac{1}{3}\sin^3 x.$

10. $\displaystyle\int \cos^4 x\,dx = \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x = \frac{3}{8}x + \frac{3}{8}\sin x\cos x + \frac{1}{4}\sin x\cos^3 x.$

11. $$\int \cos^5 x\,dx = \frac{5}{8}\sin x + \frac{5}{48}\sin 3x + \frac{1}{80}\sin 5x = \frac{4}{5}\sin x - \frac{4}{15}\sin^3 x + \frac{1}{5}\cos^4 x\sin x.$$

12. $$\int \cos^6 x\,dx = \frac{5}{16}x + \frac{15}{64}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{192}\sin 6x = \frac{5}{16}x + \frac{5}{16}\sin x\cos x + \frac{5}{24}\sin x\cos^3 x + \frac{1}{6}\sin x\cos^5 x.$$

**5.5. Интегралы вида** $\int \frac{dx}{\cos^p x}$.

**1.** $$\int \frac{dx}{\cos^p x} = \frac{\sin x}{(p-1)\cos^{p-1} x} + \frac{p-2}{p-1}\int \frac{dx}{\cos^{p-2} x}.$$

**2.** $$\int \frac{dx}{\cos^{2n} x} = \frac{\sin x}{2n-1}\left[\sec^{2n-1} x + \right.$$
$$\left. + \sum_{k=1}^{n-1} \frac{2^k (n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \sec^{2n-2k-1} x\right].$$

**3.** $$\int \frac{dx}{\cos^{2n+1} x} = \frac{\sin x}{2n}\left[\sec^{2n} x + \right.$$
$$\left. + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)} \sec^{2n-2k} x\right] +$$
$$+ \frac{(2n-1)!!}{2^n n!} \ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$$

**4.** $$\int \frac{dx}{\cos x} = \ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| = \ln\left|\operatorname{ctg}\left(\frac{\pi}{4} - \frac{x}{2}\right)\right| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|.$$

**5.** $$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x.$$

**6.** $$\int \frac{dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$$

**7.** $$\int \frac{dx}{\cos^4 x} = \frac{\sin x}{3\cos^3 x} + \frac{2}{3}\operatorname{tg} x = \frac{1}{3}\operatorname{tg}^3 x + \operatorname{tg} x.$$

**8.** $$\int \frac{dx}{\cos^5 x} = \frac{\sin x}{4\cos^4 x} + \frac{3}{8}\frac{\sin x}{\cos^2 x} + \frac{3}{8}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$$

**9.** $$\int \frac{dx}{\cos^6 x} = \frac{\sin x}{5\cos^5 x} + \frac{4}{15}\operatorname{tg}^3 x + \frac{4}{5}\operatorname{tg} x = \frac{1}{5}\operatorname{tg}^5 x + \frac{2}{3}\operatorname{tg}^3 x + \operatorname{tg} x.$$

**5.6. Интегралы вида** $\int \sin^p x \cos^q x\, dx$.

**1.** $$\int \sin^p x \cos^q x\, dx = -\frac{\sin^{p-1} x \cos^{q+1} x}{q+1} + \frac{p-1}{p+1}\int \sin^{p-2} x \cos^{q+2} x\, dx.$$

**2.** $$= -\frac{\sin^{p-1} x \cos^{q+1} x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2} x \cos^q x\, dx.$$

**3.** $$= \frac{\sin^{p+1} x \cos^{q+1} x}{p+1} + \frac{p+q+2}{p+1}\int \sin^{p+2} x \cos^q x\, dx.$$

**4.** $$= \frac{\sin^{p+1} x \cos^{q-1} x}{p+1} + \frac{q-1}{p+1}\int \sin^{p+2} x \cos^{q-2} x\, dx.$$

**5.** $$= \frac{\sin^{p+1} x \cos^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sin^p x \cos^{q-2} x\, dx.$$

**6.** $$= -\frac{\sin^{p+1} x \cos^{q+1} x}{q+1} + \frac{p+q+2}{q+1}\int \sin^p x \cos^{q+2} x\, dx.$$

**7.** $$= \frac{\sin^{p-1} x \cos^{q-1} x}{p+q}\left(\sin^2 x - \frac{q-1}{p+q-2}\right) + \\ + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2} x \cos^{q-2} x \, dx.$$

**8.** $$\int \sin^p x \cos^{2n} x \, dx = \frac{\sin^{p+1} x}{2n+p}\Bigg[\cos^{2n-1} x + \\ + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)\cos^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)}\Bigg] + \\ + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)} \int \sin^p x \, dx \qquad [p \neq -2,\ -4,\ \ldots,\ -2n].$$

**9.** $$\int \sin^p x \cos^{2n+1} x \, dx = \\ = \frac{\sin^{p+1} x}{2n+p+1}\left[\cos^{2n} x + \sum_{k=1}^{n} \frac{2^k n(n-1)\ldots(n-k+1)\cos^{2n-2k} x}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)}\right] \\ [p \neq -1,\ -3,\ \ldots,\ -(2n+1)].$$

**10.** $$\int \cos^p x \sin^{2n} x \, dx = -\frac{\cos^{p+1} x}{2n+p}\Bigg[\sin^{2n-1} x + \\ + \sum_{k=1}^{n-1} \frac{(2n-1)(2n-3)\ldots(2n-2k+1)\sin^{2n-2k-1} x}{(2n+p-2)(2n+p-4)\ldots(2n+p-2k)}\Bigg] + \\ + \frac{(2n-1)!!}{(2n+p)(2n+p-2)\ldots(p+2)} \int \cos^p x \, dx \qquad [p \neq -2,\ -4,\ \ldots,\ -2n].$$

**11.** $$\int \cos^p x \sin^{2n+1} x \, dx = -\frac{\cos^{p+1} x}{2n+p+1}\Bigg[\sin^{2n} x + \\ + \sum_{k=1}^{n} \frac{2^k n(n-1)\ldots(n-k+1)\sin^{2n-2k} x}{(2n+p-1)(2n+p-3)\ldots(2n+p-2k+1)}\Bigg] \\ [p \neq -1,\ -3,\ \ldots,\ -(2n+1)].$$

**12.** $$\int \sin x \cos^p x \, dx = -\frac{1}{p+1}\cos^{p+1} x.$$

**13.** $$\int \sin x \cos^2 x \, dx = -\frac{1}{4}\left(\frac{1}{3}\cos 3x + \cos x\right) = -\frac{\cos^3 x}{3}.$$

**14.** $$\int \sin x \cos^3 x \, dx = -\frac{\cos^4 x}{4}.$$

**15.** $$\int \sin x \cos^4 x \, dx = -\frac{\cos^5 x}{5}.$$

**16.** $$\int \sin^2 x \cos x \, dx = -\frac{1}{4}\left(\frac{1}{3}\sin 3x - \sin x\right) = \frac{\sin^3 x}{3}.$$

**17.** $$\int \sin^2 x \cos^2 x \, dx = -\frac{1}{8}\left(\frac{1}{4}\sin 4x - x\right).$$

**18.** $$\int \sin^2 x \cos^3 x\, dx = -\frac{1}{16}\left(\frac{1}{5}\sin 5x + \frac{1}{3}\sin 3x - 2\sin x\right) = \\ = \frac{\sin^3 x}{5}\left(\cos^2 x + \frac{2}{3}\right) = \frac{\sin^3 x}{5}\left(\frac{5}{3} - \sin^2 x\right).$$

**19.** $$\int \sin^2 x \cos^4 x\, dx = \frac{x}{16} + \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x - \frac{1}{192}\sin 6x.$$

**20.** $$\int \sin^3 x \cos x\, dx = \frac{1}{8}\left(\frac{1}{4}\cos 4x - \cos 2x\right) = \frac{\sin^4 x}{4} + \text{const}.$$

**21.** $$\int \sin^3 x \cos^2 x\, dx = \frac{1}{16}\left(\frac{1}{5}\cos 5x - \frac{1}{3}\cos 3x - 2\cos x\right) = \\ = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + \text{const}.$$

**22.** $$\int \sin^3 x \cos^3 x\, dx = \frac{1}{64}\left(\frac{1}{3}\cos 6x - 3\cos 2x\right).$$

**23.** $$\int \sin^3 x \cos^4 x\, dx = \frac{1}{7}\cos^3 x\left(-\frac{2}{5} - \frac{3}{5}\sin^2 x + \sin^4 x\right).$$

**24.** $$\int \sin^4 x \cos x\, dx = \frac{\sin^5 x}{5}.$$

**25.** $$\int \sin^4 x \cos^2 x\, dx = \frac{1}{16}x - \frac{1}{64}\sin 2x - \frac{1}{64}\sin 4x + \frac{1}{192}\sin 6x.$$

**26.** $$\int \sin^4 x \cos^3 x\, dx = \frac{1}{7}\sin^3 x\left(\frac{2}{5} + \frac{3}{5}\cos^2 x - \cos^4 x\right).$$

**27.** $$\int \sin^4 x \cos^4 x\, dx = \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x.$$

**28.** $$\int \sin ax \sin bx \sin cx\, dx = -\frac{1}{4}\left[\frac{\cos(a-b+c)x}{a-b+c} + \frac{\cos(b+c-a)x}{b+c-a} + \right. \\ \left. + \frac{\cos(a+b-c)x}{a+b-c} - \frac{\cos(a+b+c)x}{a+b+c}\right].$$

**29.** $$\int \sin ax \cos bx \cos cx\, dx = -\frac{1}{4}\left[\frac{\cos(a+b+c)x}{a+b+c} - \frac{\cos(b+c-a)x}{b+c-a} + \right. \\ \left. + \frac{\cos(a+b-c)x}{a+b-c} + \frac{\cos(a+c-b)x}{a+c-b}\right].$$

**30.** $$\int \cos ax \sin bx \sin cx\, dx = \\ = \frac{1}{4}\left[\frac{\sin(a+b-c)x}{a+b-c} + \frac{\sin(a+c-b)x}{a+c-b} - \frac{\sin(a+b+c)x}{a+b+c} - \frac{\sin(b+c-a)x}{b+c-a}\right].$$

**31.** $$\int \cos ax \cos bx \cos cx\, dx = \\ = \frac{1}{4}\left[\frac{\sin(a+b+c)x}{a+b+c} + \frac{\sin(b+c-a)x}{b+c-a} + \frac{\sin(a+c-b)x}{a+c-b} + \frac{\sin(a+b-c)x}{a+b-c}\right].$$

### 5.7. Интегралы вида $\int \frac{\sin^p x}{\cos^q x}\,dx$.

**1.** $$\int \frac{\sin^p x}{\cos^q x}\,dx = -\frac{\sin^{p-1} x}{(p-q)\cos^{q-1} x} + \frac{p-1}{p-q}\int \frac{\sin^{p-2} x}{\cos^q x}\,dx.$$

**2.** $$= \frac{\sin^{p+1} x}{(q-1)\cos^{q-1} x} - \frac{p-q+2}{q-1}\int \frac{\sin^p x}{\cos^{q-2} x}\,dx.$$

**3.** $$= \frac{\sin^{p-1} x}{(q-1)\cos^{q-1} x} - \frac{p-1}{q-1}\int \frac{\sin^{p-2} x}{\cos^{q-2} x}\,dx.$$

**4.** $$\int \frac{\sin^p x\,dx}{\cos^{2n+1} x} = \frac{\sin^{p+1} x}{2n} \times$$
$$\times\left[\sec^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\dots(2n-p-2k+1)}{2^k(n-1)(n-2)\dots(n-k)}\sec^{2n-2k} x\right] +$$
$$+ \frac{(2n-p-1)(2n-p-3)\dots(3-p)(1-p)}{2^n n!}\int \frac{\sin^p x}{\cos x}\,dx.$$

**5.** $$\int \frac{\sin^p x}{\cos^{2n} x}\,dx = \frac{\sin^{p+1} x}{2n-1} \times$$
$$\times\left[\sec^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\dots(2n-p-2k)}{(2n-3)(2n-5)\dots(2n-2k-1)}\sec^{2n-2k-1} x\right] +$$
$$+ \frac{(2n-p-2)(2n-p-4)\dots(-p+2)(-p)}{(2n-1)!!}\int \sin^p x\,dx.$$

**6.** $$\int \frac{\sin^{2m+1} x}{\cos^{2n+1} x}\,dx = \frac{1}{2}\sum_{\substack{k=0\\k\neq n}}^{m} (-1)^k C_m^k \frac{\cos^{2k-2n} x}{n-k} + (-1)^{n+1} C_m^n \ln|\cos x| \qquad [n \leqslant m].$$

**7.** $$= \frac{1}{2}\sum_{k=0}^{m} (-1)^k C_m^k \frac{\cos^{2k-2n} x}{n-k} \qquad [n > m].$$

**8.** $$\int \frac{\sin^{2m+1} x}{\cos^{2n} x}\,dx = \sum_{k=0}^{m} (-1)^{k+1} C_m^k \frac{\cos^{2k-2n+1} x}{2k-2n+1}.$$

**9.** $$\int \frac{\sin x}{\cos^q x}\,dx = \frac{1}{(q-1)\cos^{q-1} x}.$$

**10.** $$\int \frac{\sin^p x}{\cos^{p+2} x}\,dx = \frac{1}{p+1}\,\mathrm{tg}^{p+1} x.$$

**11.** $$\int \frac{\sin^{2n+1} x}{\cos x}\,dx = -\sum_{k=1}^{n} \frac{\sin^{2k} x}{2k} - \ln|\cos x|.$$

**12.** $$= \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2k} C_n^k \cos^{2k} x - \ln|\cos x|.$$

**13.** $$\int \frac{\sin^{2n} x\,dx}{\cos x} = -\sum_{k=1}^{n} \frac{\sin^{2k-1} x}{2k-1} + \ln\left|\mathrm{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right|.$$

**14.** $\displaystyle\int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x|.$

**15.** $\displaystyle\int \frac{\sin^2 x}{\cos x}\,dx = -\sin x + \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**16.** $\displaystyle\int \frac{\sin^3 x}{\cos x}\,dx = -\frac{1}{2}\sin^2 x - \ln|\cos x| = \frac{1}{2}\cos^2 x - \ln|\cos x|.$

**17.** $\displaystyle\int \frac{\sin^4 x}{\cos x}\,dx = -\frac{1}{3}\sin^3 x - \sin x + \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**18.** $\displaystyle\int \frac{\sin^p x}{\cos^2 x}\,dx = \frac{\sin^{p-1} x}{\cos x} - (p-1)\int \sin^{p-2} x\,dx.$

**19.** $\displaystyle\int \frac{\sin x\,dx}{\cos^2 x} = \frac{1}{\cos x}.$

**20.** $\displaystyle\int \frac{\sin^2 x\,dx}{\cos^2 x} = \operatorname{tg} x - x.$

**21.** $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^2 x} = \cos x + \frac{1}{\cos x}.$

**22.** $\displaystyle\int \frac{\sin^4 x\,dx}{\cos^2 x} = \operatorname{tg} x + \frac{1}{2}\sin x\cos x - \frac{3}{2}x.$

**23.** $\displaystyle\int \frac{\sin x\,dx}{\cos^3 x} = \frac{1}{2}\operatorname{tg}^2 x.$

**24.** $\displaystyle\int \frac{\sin^2 x\,dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} - \frac{1}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**25.** $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^3 x} = \frac{1}{2\cos^2 x} + \ln|\cos x|.$

**26.** $\displaystyle\int \frac{\sin^4 x\,dx}{\cos^3 x} = \frac{\sin x}{2\cos^2 x} + \sin x - \frac{3}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**27.** $\displaystyle\int \frac{\sin x\,dx}{\cos^4 x} = \frac{1}{3\cos^3 x}.$

**28.** $\displaystyle\int \frac{\sin^2 x\,dx}{\cos^4 x} = \frac{1}{3}\operatorname{tg}^3 x.$

**29.** $\displaystyle\int \frac{\sin^3 x\,dx}{\cos^4 x} = -\frac{1}{\cos x} + \frac{1}{3\cos^3 x}.$

**30.** $\displaystyle\int \frac{\sin^4 x\,dx}{\cos^4 x} = \frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x.$

### 5.8. Интегралы вида $\displaystyle\int \frac{\cos^q x}{\sin^p x}\,dx$.

**1.** $\displaystyle\int \frac{\cos^q x}{\sin^p x}\,dx = \frac{\cos^{q-1} x}{(q-p)\sin^{p-1} x} + \frac{q-1}{q-p}\int \frac{\cos^{q-2} x}{\sin^p x}\,dx.$

**2.** $\displaystyle\qquad = -\frac{\cos^{q+1} x}{(p-1)\sin^{p-1} x} - \frac{q-p+2}{p-1}\int \frac{\cos^q x}{\sin^{p-2} x}\,dx.$

**3.** $\displaystyle\qquad = -\frac{\cos^{q-1} x}{(p-1)\sin^{p-1} x} - \frac{q-1}{p-1}\int \frac{\cos^{q-2} x}{\sin^{p-2} x}\,dx.$

**4.** $$\int \frac{\cos^p x\,dx}{\sin^{2n+1} x} = -\frac{\cos^{p+1} x}{2n}\left[\operatorname{cosec}^{2n} x + \sum_{k=1}^{n-1} \frac{(2n-p-1)(2n-p-3)\ldots(2n-p-2k+1)}{2^k (n-1)(n-2)\ldots(n-k)} \operatorname{cosec}^{2n-2k} x\right] + \frac{(2n-p-1)(2n-p-3)\ldots(3-p)(1-p)}{2^n n!} \int \frac{\cos^p x}{\sin x}\,dx.$$

**5.** $$\int \frac{\cos^p x\,dx}{\sin^{2n} x} = -\frac{\cos^{p+1} x}{2n-1}\left[\operatorname{cosec}^{2n-1} x + \sum_{k=1}^{n-1} \frac{(2n-p-2)(2n-p-4)\ldots(2n-p-2k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \operatorname{cosec}^{2n-2k-1} x\right] + \frac{(2n-p-2)(2n-p-4)\ldots(2-p)(-p)}{(2n-1)!!} \int \cos^p x\,dx.$$

**6.** $$\int \frac{\cos^{2n+1} x}{\sin^{2m+1} x}\,dx = \frac{1}{2} \sum_{\substack{k=0 \\ k\neq m}}^{n} (-1)^k C_n^k \frac{\sin^{2k-2m} x}{m-k} + (-1)^m C_n^m \ln|\sin x| \qquad [m \leqslant n].$$

**7.** $$= \frac{1}{2} \sum_{k=0}^{n} (-1)^k C_n^k \frac{\sin^{2k-2m} x}{m-k} \qquad [m > n].$$

**8.** $$\int \frac{\cos^{2n+1} x}{\sin^{2m} x}\,dx = \sum_{k=0}^{n} (-1)^k C_n^k \frac{\sin^{2k-2m+1} x}{2k-2m+1}.$$

**9.** $$\int \frac{\cos x}{\sin^p x}\,dx = -\frac{1}{(p-1)\sin^{p-1} x}.$$

**10.** $$\int \frac{\cos^q x}{\sin^{q+2} x}\,dx = -\frac{1}{q+1} \operatorname{ctg}^{q+1} x.$$

**11.** $$\int \frac{\cos^{2n+1} x\,dx}{\sin x} = \sum_{k=1}^{n} \frac{\cos^{2k} x}{2k} + \ln|\sin x|.$$

**12.** $$= \frac{1}{2} \sum_{k=1}^{n} \frac{(-1)^k}{k} C_n^k \sin^{2k} x + \ln|\sin x|.$$

**13.** $$\int \frac{\cos^{2n} x\,dx}{\sin x} = \sum_{k=1}^{n} \frac{\cos^{2k-1} x}{2k-1} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

**14.** $$\int \frac{\cos x\,dx}{\sin x} = \ln|\sin x|.$$

**15.** $$\int \frac{\cos^2 x\,dx}{\sin x} = \cos x + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

**16.** $$\int \frac{\cos^3 x\,dx}{\sin x} = \frac{\cos^2 x}{2} + \ln|\sin x|.$$

**17.** $$\int \frac{\cos^4 x\,dx}{\sin x} = \frac{1}{3}\cos^3 x + \cos x + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

**18.** $\displaystyle\int \frac{\cos^p x}{\sin^2 x}\,dx = -\frac{\cos^{p-1} x}{\sin x} - (p-1)\int \cos^{p-2} x\,dx.$

**19.** $\displaystyle\int \frac{\cos x}{\sin^2 x}\,dx = -\frac{1}{\sin x}.$

**20.** $\displaystyle\int \frac{\cos^2 x}{\sin^2 x}\,dx = -\operatorname{ctg} x - x.$

**21.** $\displaystyle\int \frac{\cos^3 x}{\sin^2 x}\,dx = -\sin x - \frac{1}{\sin x}.$

**22.** $\displaystyle\int \frac{\cos^4 x}{\sin^2 x}\,dx = -\operatorname{ctg} x - \frac{1}{2}\sin x\cos x - \frac{3}{2}\,x.$

**23.** $\displaystyle\int \frac{\cos x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x}.$

**24.** $\displaystyle\int \frac{\cos^2 x}{\sin^3 x}\,dx = -\frac{\cos x}{2\sin^2 x} - \frac{1}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**25.** $\displaystyle\int \frac{\cos^3 x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - \ln|\sin x|.$

**26.** $\displaystyle\int \frac{\cos^4 x}{\sin^3 x}\,dx = -\frac{\cos x}{2\sin^2 x} - \cos x - \frac{3}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**27.** $\displaystyle\int \frac{\cos x}{\sin^4 x}\,dx = -\frac{1}{3\sin^3 x}.$

**28.** $\displaystyle\int \frac{\cos^2 x}{\sin^4 x}\,dx = -\frac{1}{3}\operatorname{ctg}^3 x.$

**29.** $\displaystyle\int \frac{\cos^3 x}{\sin^4 x}\,dx = \frac{1}{\sin x} - \frac{1}{3\sin^3 x}.$

**30.** $\displaystyle\int \frac{\cos^4 x}{\sin^4 x}\,dx = -\frac{1}{3}\operatorname{ctg}^3 x + \operatorname{ctg} x + x.$

### 5.9. Интегралы вида $\displaystyle\int \frac{dx}{\sin^p x\cos^q x}$.

**1.** $\displaystyle\int \frac{dx}{\sin^p x\cos^q x} = -\frac{1}{(p-1)\sin^{p-1} x\cos^{q-1} x} + \frac{p+q-2}{p-1}\int \frac{dx}{\sin^{p-2} x\cos^q x}.$

**2.** $\displaystyle\qquad = \frac{1}{(q-1)\sin^{p-1} x\cos^{q-1} x} + \frac{p+q-2}{q-1}\int \frac{dx}{\sin^p x\cos^{q-2} x}$

[см. также 5.6].

**3.** $\displaystyle\int \frac{dx}{\sin^{2m} x\cos^{2n} x} = \sum_{k=0}^{m+n-1} C_{m+n-1}^k\,\frac{\operatorname{tg}^{2k-2m+1} x}{2k-2m+1}.$

**4.** $\displaystyle\int \frac{dx}{\sin^{2m+1} x\cos^{2n+1} x} = \sum_{\substack{k=0\\k\neq m}}^{m+n} C_{m+n}^k\,\frac{\operatorname{tg}^{2k-2m} x}{2k-2m} + C_{m+n}^m\ln|\operatorname{tg} x|.$

**5.** $\displaystyle\int \frac{dx}{\sin^{2m+1} x\cos x} = -\sum_{k=1}^{m}\frac{1}{(2m-2k+2)\sin^{2m-2k+2} x} + \ln|\operatorname{tg} x|.$

**6.** $\displaystyle\int \frac{dx}{\sin^{2m} x \cos x} = -\sum_{k=1}^{m} \frac{1}{(2m-2k+1)\sin^{2m-2k+1} x} + \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**7.** $\displaystyle\int \frac{dx}{\sin x \cos^{2m+1} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+2)\cos^{2m-2k+2} x} + \ln|\operatorname{tg} x|.$

**8.** $\displaystyle\int \frac{dx}{\sin x \cos^{2m} x} = \sum_{k=1}^{m} \frac{1}{(2m-2k+1)\cos^{2m-2k+1} x} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**9.** $\displaystyle\int \frac{dx}{\sin x \cos x} = \ln|\operatorname{tg} x|.$

**10.** $\displaystyle\int \frac{dx}{\sin x \cos^2 x} = \frac{1}{\cos x} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**11.** $\displaystyle\int \frac{dx}{\sin x \cos^3 x} = \frac{1}{2\cos^2 x} + \ln|\operatorname{tg} x|.$

**12.** $\displaystyle\int \frac{dx}{\sin x \cos^4 x} = \frac{1}{\cos x} + \frac{1}{3\cos^3 x} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**13.** $\displaystyle\int \frac{dx}{\sin^2 x \cos x} = \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| - \frac{1}{\sin x}.$

**14.** $\displaystyle\int \frac{dx}{\sin^2 x \cos^2 x} = -2\operatorname{ctg} 2x.$

**15.** $\displaystyle\int \frac{dx}{\sin^2 x \cos^3 x} = \left(\frac{1}{2\cos^2 x} - \frac{3}{2}\right)\frac{1}{\sin x} + \frac{3}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**16.** $\displaystyle\int \frac{dx}{\sin^2 x \cos^4 x} = \frac{1}{3\sin x \cos^3 x} - \frac{8}{3}\operatorname{ctg} 2x.$

**17.** $\displaystyle\int \frac{dx}{\sin^3 x \cos x} = -\frac{1}{2\sin^2 x} + \ln|\operatorname{tg} x|.$

**18.** $\displaystyle\int \frac{dx}{\sin^3 x \cos^2 x} = -\frac{1}{\cos x}\left(\frac{1}{2\sin^2 x} - \frac{3}{2}\right) + \frac{3}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**19.** $\displaystyle\int \frac{dx}{\sin^3 x \cos^3 x} = -\frac{2\cos 2x}{\sin^2 2x} + 2\ln|\operatorname{tg} x|.$

**20.** $\displaystyle\int \frac{dx}{\sin^3 x \cos^4 x} = \frac{2}{\cos x} + \frac{1}{3\cos^3 x} - \frac{\cos x}{2\sin^2 x} + \frac{5}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**21.** $\displaystyle\int \frac{dx}{\sin^4 x \cos x} = -\frac{1}{\sin x} - \frac{1}{3\sin^3 x} + \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**22.** $\displaystyle\int \frac{dx}{\sin^4 x \cos^2 x} = -\frac{1}{3\cos x \sin^3 x} - \frac{8}{3}\operatorname{ctg} 2x.$

**23.** $\displaystyle\int \frac{dx}{\sin^4 x \cos^3 x} = -\frac{2}{\sin x} - \frac{1}{3\sin^3 x} + \frac{\sin x}{2\cos^2 x} + \frac{5}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

**24.** $\displaystyle\int \frac{dx}{\sin^4 x \cos^4 x} = -8\operatorname{ctg} 2x - \frac{8}{3}\operatorname{ctg}^3 2x.$

### 5.10. Интегралы вида $\int \operatorname{tg}^p x\,dx$.

**1.** $\displaystyle\int \operatorname{tg}^p x\,dx = \frac{1}{p-1}\operatorname{tg}^{p-1} x - \int \operatorname{tg}^{p-2} x\,dx.$

2. $\displaystyle\int \operatorname{tg}^{2n} x\,dx = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2n-2k+1} \operatorname{tg}^{2n-2k+1} x + (-1)^n x.$

3. $\displaystyle\int \operatorname{tg}^{2n+1} x\,dx = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2n-2k+2} \operatorname{tg}^{2n-2k+2} x + (-1)^{n+1} \ln|\cos x|.$

4. $\displaystyle\qquad = \sum_{k=1}^{n} \frac{(-1)^{n+k}}{2k} C_n^k \cos^{-2k} x + (-1)^{n+1} \ln|\cos x|.$

5. $\displaystyle\int \operatorname{tg} x\,dx = -\ln|\cos x|.$

6. $\displaystyle\int \operatorname{tg}^2 x\,dx = \operatorname{tg} x - x.$

7. $\displaystyle\int \operatorname{tg}^3 x\,dx = \frac{1}{2}\operatorname{tg}^2 x + \ln|\cos x|.$

8. $\displaystyle\int \sqrt{\operatorname{tg} x}\,dx = \frac{1}{\sqrt{2}}\left[\ln\left(\sin x + \cos x - \sqrt{\sin 2x}\right) + \arcsin(\sin x - \cos x)\right].$

**5.11. Интегралы вида** $\displaystyle\int \operatorname{ctg}^p x\,dx.$

1. $\displaystyle\int \operatorname{ctg}^p x\,dx = -\frac{1}{p-1}\operatorname{ctg}^{p-1} x - \int \operatorname{ctg}^{p-2} x\,dx.$

2. $\displaystyle\int \operatorname{ctg}^{2n} x\,dx = \sum_{k=1}^{n} \frac{(-1)^{k}}{2n-2k+1} \operatorname{ctg}^{2n-2k+1} x + (-1)^n x.$

3. $\displaystyle\int \operatorname{ctg}^{2n+1} x\,dx = \sum_{k=1}^{n} \frac{(-1)^{k}}{2n-2k+2} \operatorname{ctg}^{2n-2k+2} x + (-1)^{n} \ln|\sin x|.$

4. $\displaystyle\qquad = \sum_{k=1}^{n} \frac{(-1)^{n+k+1}}{2k} C_n^k \sin^{-2k} x + (-1)^{n} \ln|\sin x|.$

5. $\displaystyle\int \operatorname{ctg} x\,dx = \ln|\sin x|.$

6. $\displaystyle\int \operatorname{ctg}^2 x\,dx = -\operatorname{ctg} x - x.$

7. $\displaystyle\int \operatorname{ctg}^3 x\,dx = -\frac{1}{2}\operatorname{ctg}^2 x - \ln|\sin x|.$

**5.12. Интегралы вида**

$$\int \frac{A + B\cos x + C\sin x}{(a + b\cos x + c\sin x)^n (a_1 + b_1\cos x + c_1\sin x)^p}\,dx.$$

1. $\displaystyle\int \frac{A + B\cos x + C\sin x}{(a + b\cos x + c\sin x)^n}\,dx =$

$$= \frac{(Bc - Cb) + (Ac - Ca)\cos x - (Ab - Ba)\sin x}{(n-1)(a^2 - b^2 - c^2)(a + b\cos x + c\sin x)^{n-1}} + \frac{1}{(n-1)(a^2 - b^2 - c^2)} \times$$

$$\times \int \frac{(n-1)(Aa - Bb - Cc) - (n-2)[(Ab - Ba)\cos x - (Ac - Ca)\sin x]}{(a + b\cos x + c\sin x)^{n-1}}\,dx$$

$$[n \geqslant 2;\ a^2 \neq b^2 + c^2].$$

**2.** $$= \frac{Cb - Bc + Ca\cos x - Ba\sin x}{(n-1)a(a + b\cos x + c\sin x)^n} + \left[\frac{A}{a} + \frac{n(Bb + Cc)}{(n-1)a^2}\right] \times$$
$$\times (-c\cos x + b\sin x)\frac{(n-1)!}{(2n-1)!!}\sum_{k=0}^{n-1}\frac{(2n-2k-3)!!}{(n-k-1)!a^k}\,\frac{1}{(a + b\cos x + c\sin x)^{n-k}}$$
$$[n \geqslant 2;\ a^2 = b^2 + c^2].$$

**3.** $$\int \frac{A + B\cos x + C\sin x}{a + b\cos x + c\sin x}\,dx = \frac{Bc - Cb}{b^2 + c^2}\ln|a + b\cos x + c\sin x| +$$
$$+ \frac{Bb + Cc}{b^2 + c^2}\,x + \left(A - \frac{Bb + Cc}{b^2 + c^2}\,a\right)\int \frac{dx}{a + b\cos x + c\sin x}.$$

**4.** $$\int \frac{dx}{(a + b\cos x + c\sin x)^n} = \int \frac{d(x - \alpha)}{[a + r\cos(x - \alpha)]^n}$$
$$[r\cos\alpha = b,\ r\sin\alpha = c].$$

**5.** $$\int \frac{dx}{a + b\cos x + c\sin x} = \frac{2}{\sqrt{a^2 - b^2 - c^2}}\operatorname{arctg}\frac{(a-b)\operatorname{tg}\dfrac{x}{2} + c}{\sqrt{a^2 - b^2 - c^2}}$$
$$[a^2 > b^2 + c^2].$$

**6.** $$= \frac{1}{\sqrt{b^2 + c^2 - a^2}}\ln\left|\frac{(a-b)\operatorname{tg}\dfrac{x}{2} + c - \sqrt{b^2 + c^2 - a^2}}{(a-b)\operatorname{tg}\dfrac{x}{2} + c + \sqrt{b^2 + c^2 - a^2}}\right|$$
$$[a^2 < b^2 + c^2].$$

**7.** $$= \frac{1}{c}\ln\left|a + c\operatorname{tg}\frac{x}{2}\right| \qquad [a = b].$$

**8.** $$= -\frac{2}{c + (a-b)\operatorname{tg}\dfrac{x}{2}} \qquad [a^2 = b^2 + c^2].$$

**9.** $$\int \frac{dx}{(a\cos x + b\sin x)^n} = \frac{1}{\sqrt{(a^2 + b^2)^n}}\int \frac{dx}{\sin^n\left(x + \operatorname{arctg}\dfrac{a}{b}\right)} \qquad [\text{см. } 5.5].$$

**10.** $$\int \frac{A + B\cos x + C\sin x}{a\cos x + b\sin x}\,dx = \frac{Ba + Cb}{a^2 + b^2}\,x +$$
$$+ \frac{Bb - Ca}{a^2 + b^2}\ln\left|\sin\left(x + \operatorname{arctg}\frac{a}{b}\right)\right| + \frac{A}{\sqrt{a^2 + b^2}}\ln\left|\operatorname{tg}\left[\frac{1}{2}\left(x + \operatorname{arctg}\frac{a}{b}\right)\right]\right|.$$

**11.** $$\int \frac{\sin x}{a\cos x + b\sin x}\,dx = \frac{1}{a^2 + b^2}(bx - a\ln|a\cos x + b\sin x|).$$

**12.** $$\int \frac{\cos x}{a\cos x + b\sin x}\,dx = \frac{1}{a^2 + b^2}(ax + b\ln|a\cos x + b\sin x|).$$

**13.** $$\int \frac{dx}{(a\cos x + b\sin x)^2} = -\frac{1}{a^2 + b^2}\cdot\frac{a\sin x - b\cos x}{a\cos x + b\sin x}.$$

**14.** $$\int \frac{dx}{a+b\sin x} = \frac{2}{\sqrt{a^2-b^2}} \operatorname{arctg} \frac{a \operatorname{tg}\dfrac{x}{2}+b}{\sqrt{a^2-b^2}} \qquad [a^2>b^2].$$

**15.** $$= \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{a \operatorname{tg}\dfrac{x}{2}+b-\sqrt{b^2-a^2}}{a \operatorname{tg}\dfrac{x}{2}+b+\sqrt{b^2-a^2}} \right| \qquad [a^2<b^2].$$

**16.** $$\int \frac{dx}{a+b\cos x} = \frac{2}{\sqrt{a^2-b^2}} \operatorname{arctg} \frac{\sqrt{a^2-b^2}\,\operatorname{tg}\dfrac{x}{2}}{a+b} \qquad [a^2>b^2].$$

**17.** $$= \frac{1}{\sqrt{b^2-a^2}} \ln \left| \frac{\sqrt{b^2-a^2}\,\operatorname{tg}\dfrac{x}{2}+a+b}{\sqrt{b^2-a^2}\,\operatorname{tg}\dfrac{x}{2}-a-b} \right| \qquad [a^2<b^2].$$

**18.** $$\int \frac{A+B\cos x+C\sin x}{(1\pm\sin x)^n}\,dx = -\frac{1}{2^{n-1}}\left[2C\sum_{k=0}^{n-2} C_{n-2}^k \frac{\operatorname{tg}^{2k+1}\left(\dfrac{\pi}{4}\mp\dfrac{x}{2}\right)}{2k+1} \pm \right.$$
$$\left. \pm (A\mp C)\sum_{k=0}^{n-1} C_{n-1}^k \frac{\operatorname{tg}^{2k+1}\left(\dfrac{\pi}{4}\mp\dfrac{x}{2}\right)}{2k+1}\right] \mp \frac{B}{(n-1)(1\pm\sin x)^{n-1}} \qquad [n\geqslant 2].$$

**19.** $$= \pm Cx + (A\mp C)\operatorname{tg}\left(\frac{\pi}{4}\mp\frac{x}{2}\right) \pm B\ln(1\pm\sin x) \qquad [n=1].$$

**20.** $$\int \frac{A+B\cos x+C\sin x}{(1\pm\cos x)^n}\,dx = \frac{1}{2^{n-1}}\left[2B\sum_{k=0}^{n-2} C_{n-2}^k \frac{\operatorname{tg}^{\pm(2k+1)}\dfrac{x}{2}}{2k+1} \pm \right.$$
$$\left. \pm (A\mp B)\sum_{k=0}^{n-1} C_{n-1}^k \frac{\operatorname{tg}^{\pm(2k+1)}\dfrac{x}{2}}{2k+1}\right] \pm \frac{C}{(n-1)(1\pm\cos x)^{n-1}} \qquad [n\geqslant 2].$$

**21.** $$= \pm Bx \pm (A\mp B)\operatorname{tg}^{\pm 1}\frac{x}{2} \mp C\ln(1\pm\cos x) \qquad [n=1].$$

**22.** $$\int \frac{A+B\cos x+C\sin x}{\sin x\,(a+b\sin x)}\,dx =$$
$$= \frac{A}{a}\ln\left|\operatorname{tg}\frac{x}{2}\right| - \frac{B}{a}\ln\left|\frac{a+b\sin x}{\sin x}\right| + \frac{Ca-Ab}{a}\int\frac{dx}{a+b\sin x}.$$

**23.** $$\int \frac{A+B\cos x+C\sin x}{\sin x\,(a+b\cos x)}\,dx =$$
$$= \frac{1}{a^2-b^2}\left[(Aa-Bb)\ln\left|\operatorname{tg}\frac{x}{2}\right| + (Ab-Ba)\ln\left|\frac{a+b\cos x}{\sin x}\right|\right] + C\int\frac{dx}{a+b\cos x}.$$

**24.** $$\int \frac{A+B\cos x+C\sin x}{\sin x\,(1\pm\cos x)}\,dx =$$
$$= \frac{A\pm B}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right| - \frac{B\mp A}{2}\,\frac{1}{1\pm\cos x} \pm C\left(\operatorname{tg}\frac{x}{2}\right)^{\pm 1}.$$

**25.** $$\int \frac{A + B\cos x + C\sin x}{\cos x\,(a + b\sin x)}\,dx = \frac{1}{a^2 - b^2}\left[(Aa - Cb)\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| - (Ab - Ca)\ln\left|\frac{a + b\sin x}{\cos x}\right|\right] + B\int \frac{dx}{a + b\sin x}.$$

**26.** $$\int \frac{A + B\cos x + C\sin x}{\cos x\,(1 \pm \sin x)}\,dx = \frac{A \pm C}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + \frac{A \mp C}{2(1 \pm \sin x)} + B\operatorname{tg}\left(\frac{\pi}{4} \pm \frac{x}{2}\right).$$

**27.** $$\int \frac{A + B\cos x + C\sin x}{\cos x\,(a + b\cos x)}\,dx = \frac{A}{a}\ln\left|\operatorname{tg}\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + \frac{C}{a}\ln\left|\frac{a + b\cos x}{\cos x}\right| + \left(B - \frac{Ab}{a}\right)\int \frac{dx}{a + b\cos x}.$$

**28.** $$\int \frac{A + B\cos x + C\sin x}{(a_1 + b_1\cos x + c_1\sin x)(a_2 + b_2\cos x + c_2\sin x)}\,dx = A_0\ln\left|\frac{a_1 + b_1\cos x + c_1\sin x}{a_2 + b_2\cos x + c_2\sin x}\right| + A_1\int \frac{dx}{a_1 + b_1\cos x + c_1\sin x} + A_2\int \frac{dx}{a_2 + b_2\cos x + c_2\sin x},$$

где

$$A_0 = \frac{\begin{vmatrix} A & B & C \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_1 = \frac{\begin{vmatrix} \begin{vmatrix} B & C \\ b_1 & c_1 \end{vmatrix} & \begin{vmatrix} A & C \\ a_1 & c_1 \end{vmatrix} & \begin{vmatrix} B & A \\ b_1 & a_1 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2},$$

$$A_2 = \frac{\begin{vmatrix} \begin{vmatrix} C & B \\ c_2 & b_2 \end{vmatrix} & \begin{vmatrix} C & A \\ c_2 & a_2 \end{vmatrix} & \begin{vmatrix} A & B \\ a_2 & b_2 \end{vmatrix} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 - \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2} \qquad \left[\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}^2 + \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}^2 \neq \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}^2\right].$$

**5.13.** Интегралы вида $\displaystyle\int \frac{dx}{(a+b\sin^2 x)^n}$.

**1.** $\displaystyle\int \frac{dx}{a+b\sin^2 x} = \frac{\operatorname{sign} a}{\sqrt{a(a+b)}} \operatorname{arctg}\left(\sqrt{\frac{a+b}{a}}\,\operatorname{tg} x\right)$ $[b/a > -1]$.

**2.** $\displaystyle = \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth}\left(\sqrt{-\frac{a+b}{a}}\,\operatorname{tg} x\right)$ $[b/a < -1;\ \sin^2 x < -a/b]$.

**3.** $\displaystyle = \frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arcth}\left(\sqrt{-\frac{a+b}{a}}\,\operatorname{tg} x\right)$ $[b/a < -1;\ \sin^2 x > -a/b]$.

**4.** $\displaystyle\int \frac{dx}{1+\sin^2 x} = \frac{1}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x)$.

**5.** $\displaystyle\int \frac{dx}{1-\sin^2 x} = \operatorname{tg} x$.

**6.** $\displaystyle\int \frac{dx}{(a+b\sin^2 x)^2} = \frac{1}{2a(a+b)}\left[(2a+b)\int \frac{dx}{a+b\sin^2 x} + \frac{b\sin x\cos x}{a+b\sin^2 x}\right]$.

**7.** $\displaystyle\int \frac{dx}{(1+\sin^2 x)^2} = \frac{1}{4}\left[\frac{3}{\sqrt{2}} \operatorname{arctg}(\sqrt{2}\,\operatorname{tg} x) + \frac{\sin x\cos x}{1+\sin^2 x}\right]$.

**8.** $$\int \frac{dx}{(a+b\sin^2 x)^3} = \frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\operatorname{arctg}(p\operatorname{tg} x) + \right.$$
$$\left. + \left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\operatorname{tg} x}{1+p^2\operatorname{tg}^2 x} + \left(1-\frac{2}{p^2}-\frac{1}{p^2}\operatorname{tg}^2 x\right)\frac{2p\operatorname{tg} x}{(1+p^2\operatorname{tg}^2 x)^2}\right]$$
$[p^2 = 1 + b/a > 0]$.

**9.** $$= \frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{Arth}(q\operatorname{tg} x) + \right.$$
$$\left. + \left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\operatorname{tg} x}{1-q^2\operatorname{tg}^2 x} + \left(1+\frac{2}{q^2}+\frac{1}{q^2}\operatorname{tg}^2 x\right)\frac{2q\operatorname{tg} x}{(1-q^2\operatorname{tg}^2 x)^2}\right]$$
$[q^2 = -1 - b/a > 0;\ \sin^2 x < -a/b;$
при $\sin^2 x > -a/b$ следует $\operatorname{Arth}(q\operatorname{tg} x)$ заменить на $\operatorname{Arcth}(q\operatorname{tg} x)]$.

**5.14.** Интегралы вида $\displaystyle\int \frac{dx}{(a+b\cos^2 x)^n}$.

**1.** $\displaystyle\int \frac{dx}{a+b\cos^2 x} = -\frac{\operatorname{sign} a}{\sqrt{a(a+b)}} \operatorname{arctg}\left(\sqrt{\frac{a+b}{a}}\,\operatorname{ctg} x\right)$ $[b/a > -1]$.

**2.** $\displaystyle = -\frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arth}\left(\sqrt{-\frac{a+b}{a}}\,\operatorname{ctg} x\right)$ $[b/a < -1;\ \cos^2 x < -a/b]$.

**3.** $\displaystyle = -\frac{\operatorname{sign} a}{\sqrt{-a(a+b)}} \operatorname{Arcth}\left(\sqrt{-\frac{a+b}{a}}\,\operatorname{ctg} x\right)$ $[b/a < -1;\ \cos^2 x > -a/b]$.

**4.** $\displaystyle\int \frac{dx}{1+\cos^2 x} = -\frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{ctg} x).$

**5.** $\displaystyle\int \frac{dx}{1-\cos^2 x} = -\operatorname{ctg} x.$

**6.** $\displaystyle\int \frac{dx}{(a+b\cos^2 x)^2} = \frac{1}{2a(a+b)}\left[(2a+b)\int \frac{dx}{a+b\cos^2 x} + \frac{b\sin x\cos x}{a+b\cos^2 x}\right].$

**7.** $\displaystyle\int \frac{dx}{(1+\cos^2 x)^2} = \frac{1}{4}\left[-\frac{3}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{ctg} x) - \frac{\sin x\cos x}{1+\cos^2 x}\right].$

**8.** $$\int \frac{dx}{(a+b\cos^2 x)^3} = -\frac{1}{8pa^3}\left[\left(3+\frac{2}{p^2}+\frac{3}{p^4}\right)\operatorname{arctg}(p\operatorname{ctg} x) + \right.$$
$$\left. + \left(3+\frac{2}{p^2}-\frac{3}{p^4}\right)\frac{p\operatorname{ctg} x}{1+p^2\operatorname{ctg}^2 x} + \left(1-\frac{2}{p^2}-\frac{1}{p^2}\operatorname{ctg}^2 x\right)\frac{2p\operatorname{ctg} x}{(1+p^2\operatorname{ctg}^2 x)^2}\right]$$
$$[p^2 = 1 + b/a > 0].$$

**9.** $$= -\frac{1}{8qa^3}\left[\left(3-\frac{2}{q^2}+\frac{3}{q^4}\right)\operatorname{Arth}(q\operatorname{ctg} x) + \right.$$
$$\left. + \left(3-\frac{2}{q^2}-\frac{3}{q^4}\right)\frac{q\operatorname{ctg} x}{1-q^2\operatorname{ctg}^2 x} + \left(1+\frac{2}{q^2}+\frac{1}{q^2}\operatorname{ctg}^2 x\right)\frac{2q\operatorname{ctg} x}{(1-q^2\operatorname{ctg}^2 x)^2}\right]$$
$[q^2 = -1 - b/a > 0;\ \cos^2 x < -a/b;$ при $\cos^2 x > -a/b$ следует $\operatorname{Arth}(q\operatorname{ctg} x)$ заменить на $\operatorname{Arcth}(q\operatorname{ctg} x)]$.

## 5.15. Интегралы вида $\displaystyle\int \frac{A\cos^2 x + B\sin x\cos x + C\sin^2 x}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}\,dx.$

**1.** $$\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x} =$$
$$= \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2c\operatorname{tg} x + b - \sqrt{b^2-4ac}}{2c\operatorname{tg} x + b + \sqrt{b^2-4ac}}\right| \qquad [b^2 > 4ac].$$

**2.** $$= \frac{2}{\sqrt{4ac-b^2}}\operatorname{arctg}\frac{2c\operatorname{tg} x + b}{\sqrt{4ac-b^2}} \qquad [b^2 < 4ac].$$

**3.** $$= -\frac{2}{2c\operatorname{tg} x + b} \qquad [b^2 = 4ac].$$

**4.** $$\int \frac{A\cos^2 x + B\sin x\cos x + C\sin^2 x}{a\cos^2 x + b\sin x\cos x + c\sin^2 x}\,dx =$$
$$= \frac{1}{b^2+(a-c)^2}\left\{[Bb + (A-C)(a-c)]x + \right.$$
$$+ \frac{1}{2}[(A-C)b - B(a-c)]\ln|a\cos^2 x + b\sin x\cos x + c\sin^2 x| +$$
$$\left. + \left[\frac{1}{2}(A+C)b^2 - \frac{1}{2}Bb(a+c) + (Ca-Ac)(a-c)\right]f(x)\right\},$$
где
$$f(x) = \frac{1}{\sqrt{b^2-4ac}}\ln\left|\frac{2c\operatorname{tg} x + b - \sqrt{b^2-4ac}}{2c\operatorname{tg} x + b + \sqrt{b^2-4ac}}\right| \qquad [b^2 > 4ac],$$
$$= \frac{1}{\sqrt{4ac-b^2}}\operatorname{arctg}\frac{2c\operatorname{tg} x - b}{\sqrt{4ac-b^2}} \qquad [b^2 < 4ac],$$

$$= -\frac{2}{2c\ \operatorname{tg} x + b} \qquad [b^2 = 4ac].$$

**5.** $$\int \frac{dx}{(a\cos^2 x + b\sin x\cos x + c\sin^2 x)^2} =$$
$$= \frac{1}{4ac - b^2} \cdot \frac{b\cos 2x + (c-a)\sin 2x}{a\cos^2 x + b\sin x\cos x + c\sin^2 x} +$$
$$+ \frac{2(a+c)}{4ac - b^2}\int \frac{dx}{a\cos^2 x + b\sin x\cos x + c\sin^2 x} \qquad [b^2 \neq 4ac].$$

**6.** $$= 16a^2 \int \frac{dx}{(2a\cos x + b\sin x)^4} \qquad [b^2 = 4ac].$$

**5.16. Интегралы вида** $\displaystyle\int \frac{A\operatorname{tg} x + B}{a\operatorname{tg}^2 x + b\operatorname{tg} x + c}\,dx.$

**1.** $$\int \frac{\operatorname{tg} x\,dx}{\operatorname{tg} x + a} = \frac{1}{1+a^2}\left[x - a\ln|a\cos x + \sin x|\right].$$

**2.** $$\int \frac{\operatorname{tg} x - a}{\operatorname{tg} x + a}\,dx = \frac{1-a^2}{1+a^2}\,x - \frac{2a}{1+a^2}\ln|\sin(x + \operatorname{arctg} a)|.$$

**3.** $$\int \frac{dx}{\operatorname{tg}^2 x + a^2} = \frac{x}{a^2-1} - \frac{1}{(a^2-1)a}\operatorname{arctg}\frac{\operatorname{tg} x}{a}.$$

**4.** $$\int \frac{dx}{\operatorname{tg}^2 x - a^2} = -\frac{x}{1+a^2} + \frac{1}{2(1+a^2)a}\ln\left|\frac{\operatorname{tg} x - a}{\operatorname{tg} x + a}\right|.$$

**5.** $$\int \frac{dx}{1 + \operatorname{tg}^2 x} = \frac{x}{2} + \frac{1}{4}\sin 2x.$$

**6.** $$\int \frac{\operatorname{tg} x\,dx}{\operatorname{tg}^2 x + a^2} = \frac{\ln(a^2\cos^2 x + \sin^2 x)}{2(1-a^2)}.$$

**5.17. Интегралы вида** $\displaystyle\int \sin(ax+b)\sin(cx+d)\,dx,$
$\displaystyle\int \cos(ax+b)\cos(cx+d)\,dx, \int \sin(ax+b)\cos(cx+d)\,dx.$

**1.** $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b).$$

**2.** $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b).$$

**3.** $$\int \sin(ax+b)\sin(cx+d)\,dx = \frac{\sin[(a-c)x + b - d]}{2(a-c)} - \frac{\sin[(a+c)x + b + d]}{2(a+c)}.$$

**4.** $$\int \cos(ax+b)\cos(cx+d)\,dx = \frac{\sin[(a-c)x + b - d]}{2(a-c)} + \frac{\sin[(a+c)x + b + d]}{2(a+c)}.$$

**5.** $$\int \sin(ax+b)\cos(cx+d)\,dx =$$
$$= -\frac{\cos[(a-c)x + b - d]}{2(a-c)} - \frac{\cos[(a+c)x + b + d]}{2(a+c)}.$$

**6.** $$\int \sin(ax+b)\sin(ax+d)\,dx = \frac{x}{2}\cos(b-d) - \frac{\sin(2ax+b+d)}{4a}.$$

**7.** $$\int \cos(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\cos(b-d) + \frac{\sin(2ax+b+d)}{4a}.$$

**8.** $\displaystyle\int \sin(ax+b)\cos(ax+d)\,dx = \frac{x}{2}\sin(b-d) - \frac{\cos(2ax+b+d)}{4a}.$

## 5.18. Интегралы вида $\int \sin^p x \sin ax\,dx$.

**1.** $\displaystyle\int \sin^p x \sin ax\,dx = \frac{\sin^p x\cos ax}{p+a} + \frac{p}{p+a}\int \sin^{p-1} x\cos(a-1)x\,dx.$

**2.** $$= -\frac{\sin^p x\cos ax}{p+a} + \frac{p\sin^{p-1} x\sin(a-1)x}{(p+a)(p+a-2)} - \\ - \frac{p(p-1)}{(p+a)(p+a-2)}\int \sin^{p-2} x\sin(a-2)x\,dx.$$

**3.** $$\int \sin^p x\sin 2nx\,dx = 2n\left\{\frac{\sin^{p+2} x}{p+2} + \right. \\ \left. + \sum_{k=1}^{n-1}(-1)^k\frac{(4n^2-2^2)(4n^2-4^2)\ldots[4n^2-(2k)^2]}{(2k+1)!(2k+p+2)}\sin^{2k+p+2} x\right\} \\ [p \neq -2,\ -4,\ \ldots,\ -2n].$$

**4.** $$\int \sin^p x\sin(2n+1)x\,dx = (2n+1)\left\{\int \sin^{p+1} x\,dx + \right. \\ + \sum_{k=1}^{n}(-1)^k\frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\times \\ \left. \times\int \sin^{2k+p+1} x\,dx\right\}.$$

**5.** $\displaystyle\int \sin^{p-1} x\sin(p+1)x\,dx = \frac{1}{p}\sin^p x\sin px.$

**6.** $\displaystyle\int \sin^{p-1} x\sin\left[(p+1)\left(\frac{\pi}{2}-x\right)\right]dx = \frac{1}{p}\sin^p x\cos p\left(\frac{\pi}{2}-x\right).$

**7.** $\displaystyle\int \frac{\sin ax}{\sin^p x}\,dx = 2\int \frac{\cos(a-1)x}{\sin^{p-1} x}\,dx + \int \frac{\sin(a-2)x}{\sin^p x}\,dx.$

**8.** $\displaystyle\int \frac{\sin 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1}.$

**9.** $\displaystyle\int \frac{\sin(2n+1)x}{\sin x}\,dx = \sum_{k=1}^{n}\frac{\sin 2kx}{k} + x.$

**10.** $\displaystyle\int \frac{\sin 2x}{\sin^n x}\,dx = -\frac{2}{(n-1)\sin^{n-2} x}.$

**11.** $\displaystyle\int \frac{\sin 2x}{\sin^2 x}\,dx = 2\ln|\sin x|.$

**12.** $\displaystyle\int \frac{\sin 3x\,dx}{\sin x} = x + \sin 2x.$

**13.** $\displaystyle\int \frac{\sin 3x}{\sin^2 x}\,dx = 3\ln\left|\operatorname{tg}\frac{x}{2}\right| + 4\cos x.$

**14.** $\displaystyle\int \frac{\sin 3x}{\sin^3 x}\,dx = -3\operatorname{ctg} x - 4x.$

## 5.19. Интегралы вида $\int \sin^p x \cos ax\,dx$.

**1.** $$\int \sin^p x \cos ax\,dx = \frac{\sin^p x \sin ax}{p+a} - \frac{p}{p+a}\int \sin^{p-1} x \sin(a-1)x\,dx.$$

**2.** $$= \frac{\sin^p x \sin ax}{p+a} + \frac{p\sin^{p-1} x \cos(a-1)x}{(p+a)(p+a-2)} - \frac{p(p-1)}{(p+a)(p+a-2)}\int \sin^{p-2} x \cos(a-2)x\,dx.$$

**3.** $$\int \sin^p x \cos 2nx\,dx = \int \sin^p x\,dx + \sum_{k=1}^{n}(-1)^k \frac{4n^2(4n^2-2^2)\ldots[4n^2-(2k-2)^2]}{(2k)!}\int \sin^{2k+p} x\,dx.$$

**4.** $$\int \sin^p x \cos(2n+1)x\,dx = \frac{\sin^{p+1} x}{p+1} + \sum_{k=1}^{n}(-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \times \sin^{2k+p+1} x \qquad [p \neq -1,\ -3,\ -5,\ \ldots,\ -(2n+1)].$$

**5.** $$\int \sin^{p-1} x \cos(p+1)x\,dx = \frac{1}{p}\sin^p x \cos px.$$

**6.** $$\int \sin^{p-1} x \cos\left[(p+1)\left(\frac{\pi}{2}-x\right)\right]dx = -\frac{1}{p}\sin^p x \sin p\left(\frac{\pi}{2}-x\right).$$

**7.** $$\int \frac{\cos ax}{\sin^p x}\,dx = -2\int \frac{\sin(a-1)x}{\sin^{p-1} x}\,dx + \int \frac{\cos(a-2)x}{\sin^p x}\,dx.$$

**8.** $$\int \frac{\cos 2nx}{\sin x}\,dx = 2\sum_{k=1}^{n}\frac{\cos(2k-1)x}{2k-1} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

**9.** $$\int \frac{\cos(2n+1)x}{\sin x}\,dx = \sum_{k=1}^{n}\frac{\cos 2kx}{k} + \ln|\sin x|.$$

**10.** $$\int \frac{\cos 2x}{\sin x}\,dx = 2\cos x + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

**11.** $$\int \frac{\cos 2x}{\sin^2 x}\,dx = -\operatorname{ctg} x - 2x.$$

**12.** $$\int \frac{\cos 2x}{\sin^3 x}\,dx = -\frac{\cos x}{2\sin^2 x} - \frac{3}{2}\ln\left|\operatorname{tg}\frac{x}{2}\right|.$$

**13.** $$\int \frac{\cos 3x}{\sin^n x}\,dx = \frac{4}{(n-3)\sin^{n-3} x} - \frac{1}{(n-1)\sin^{n-1} x}.$$

**14.** $$\int \frac{\cos 3x}{\sin x}\,dx = -2\sin^2 x + \ln|\sin x|.$$

**15.** $$\int \frac{\cos 3x}{\sin^3 x}\,dx = -\frac{1}{2\sin^2 x} - 4\ln|\sin x|.$$

### 5.20. Интегралы вида $\int \cos^p x \sin ax\, dx$.

1. $$\int \cos^p x \sin ax\, dx = -\frac{\cos^p x \cos ax}{p+a} + \frac{p}{p+a}\int \cos^{p-1} x \sin(a-1)x\, dx.$$

2. $$\int \cos^p x \sin 2nx\, dx = (-1)^n \left\{ \frac{\cos^{p+2} x}{p+2} + \sum_{k=1}^{n-1} (-1)^k \frac{(4n^2-2^2)(4n^2-4^2)\ldots[4n^2-(2k)^2]}{(2k+1)!(2k+p+2)} \cos^{2k+p+2} x \right\}$$ $[p \neq -2,\ -4,\ \ldots,\ -2n]$.

3. $$\int \cos^p x \sin(2n+1)x\, dx = (-1)^{n+1} \left\{ \frac{\cos^{p+1} x}{p+1} + \sum_{k=1}^{n} (-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!(2k+p+1)} \times \cos^{2k+p+1} x \right\}$$ $[p \neq -1,\ -3,\ -5,\ \ldots,\ -(2n+1)]$.

4. $$\int \cos^{p-1} x \sin(p+1)x\, dx = -\frac{1}{p}\cos^p x \cos px.$$

5. $$\int \frac{\sin ax}{\cos^p x}\, dx = 2\int \frac{\sin(a-1)x}{\cos^{p-1} x}\, dx - \int \frac{\sin(a-2)x}{\cos^p x}\, dx.$$

6. $$\int \frac{\sin(2n+1)x}{\cos x}\, dx = \sum_{k=1}^{n} (-1)^{n+k+1} \frac{\cos 2kx}{k} + (-1)^{n+1} \ln|\cos x|.$$

7. $$\int \frac{\sin 2nx}{\cos x}\, dx = 2\sum_{k=1}^{n} (-1)^{n+k+1} \frac{\cos(2k-1)x}{2k-1}.$$

8. $$\int \frac{\sin 2x}{\cos^n x}\, dx = \frac{2}{(n-2)\cos^{n-2} x}.$$

9. $$\int \frac{\sin 2x}{\cos^2 x}\, dx = -2\ln|\cos x|.$$

10. $$\int \frac{\sin 3x}{\cos^n x}\, dx = \frac{4}{(n-3)\cos^{n-3} x} - \frac{1}{(n-1)\cos^{n-1} x}.$$

11. $$\int \frac{\sin 3x}{\cos x}\, dx = 2\sin^2 x + \ln|\cos x|.$$

12. $$\int \frac{\sin 3x}{\cos^3 x}\, dx = -\frac{1}{2\cos^2 x} - 4\ln|\cos x|.$$

### 5.21. Интегралы вида $\int \cos^p x \cos ax\, dx$.

1. $$\int \cos^p x \cos ax\, dx = \frac{\cos^p x \sin ax}{p+a} + \frac{p}{p+a}\int \cos^{p-1} x \cos(a-1)x\, dx.$$

2. $$\int \cos^p x \cos 2nx\, dx = (-1)^n \left\{ \int \cos^p x\, dx + \sum_{k=1}^{n} (-1)^k \frac{4n^2[2n^2-2^2]\ldots[4n^2-(2k-2)^2]}{(2k)!} \int \cos^{2k+p} x\, dx \right\}.$$

3. $\displaystyle\int \cos^p x \cos(2n+1)x\,dx = (-1)^n(2n+1)\Bigg\{\int \cos^{p+1} x\,dx +$

$\displaystyle + \sum_{k=1}^{n}(-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k+1)!}\times$

$\displaystyle\times\int \cos^{2k+p+1} x\,dx\Bigg\}.$

4. $\displaystyle\int \cos^{p-1} x\cos(p+1)x\,dx = \frac{1}{p}\cos^p x\sin px.$

5. $\displaystyle\int \frac{\cos ax}{\cos^p x}\,dx = 2\int\frac{\cos(a-1)x}{\cos^{p-1}x}\,dx - \int\frac{\cos(a-2)x}{\cos^p x}\,dx.$

6. $\displaystyle\int \frac{\cos 2nx}{\cos x}\,dx = 2\sum_{k=1}^{n}(-1)^{n-k}\frac{\sin(2k-1)x}{2k-1} + (-1)^n\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

7. $\displaystyle\int \frac{\cos(2n+1)x}{\cos x}\,dx = \sum_{k=1}^{n}(-1)^{n-k}\frac{\sin 2kx}{k} + (-1)^n x.$

8. $\displaystyle\int \frac{\cos 2x}{\cos x}\,dx = 2\sin x - \ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

9. $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx = 2x - \operatorname{tg} x.$

10. $\displaystyle\int \frac{\cos 2x}{\cos^3 x}\,dx = -\frac{\sin x}{2\cos^2 x} + \frac{3}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

11. $\displaystyle\int \frac{\cos 3x}{\cos x}\,dx = \sin 2x - x.$

12. $\displaystyle\int \frac{\cos 3x}{\cos^2 x}\,dx = 4\sin x - 3\ln\left|\operatorname{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|.$

13. $\displaystyle\int \frac{\cos 3x}{\cos^3 x}\,dx = 4x - 3\operatorname{tg} x.$

**5.22.** Интегралы вида $\displaystyle\int\frac{\sin^m x}{\sin nx}\,dx,\ \int\frac{\cos^m x}{\cos nx}\,dx.$

1. $\displaystyle\int \frac{\sin^m x\,dx}{\sin(2n+1)x} = \frac{1}{2n+1}\sum_{k=0}^{2n}(-1)^{n+k}\cos^m\left[\frac{2k+1}{2(2n+1)}\pi\right]\times$

$\displaystyle\times\ln\left|\frac{\sin\left[\dfrac{k-n}{2(2n+1)}\pi+\dfrac{x}{2}\right]}{\sin\left[\dfrac{k+n+1}{2(2n+1)}\pi-\dfrac{x}{2}\right]}\right| \qquad [m\leqslant 2n].$

2. $\displaystyle\int \frac{\sin^{2m} x}{\sin 2nx}\,dx =$

$\displaystyle = \frac{(-1)^n}{2n}\left[\ln|\cos x| + \sum_{k=1}^{n-1}(-1)^k\cos^{2m}\frac{k\pi}{2n}\ln\left|\cos^2 x - \sin^2\frac{k\pi}{2n}\right|\right] \qquad [m\leqslant n].$

3. $\displaystyle\int \frac{\sin^{2m+1} x}{\sin 2nx}\,dx = \frac{(-1)^n}{2n}\Bigg[\ln\left|\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| +$

$\displaystyle + \sum_{k=1}^{n-1}(-1)^k\cos^{2m+1}\frac{k\pi}{2n}\ln\left|\operatorname{tg}\left(\frac{n+k}{4n}\pi-\frac{x}{2}\right)\operatorname{tg}\left(\frac{n-k}{4n}\pi-\frac{x}{2}\right)\right|\Bigg] \qquad [m<n].$

**4.** $$\int \frac{\cos^m x}{\cos nx}\,dx = \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k \cos^m \frac{2k+1}{2n}\pi \ln\left|\frac{\sin\left(\frac{2k+1}{4n}\pi+\frac{x}{2}\right)}{\sin\left(\frac{2k+1}{4n}\pi-\frac{x}{2}\right)}\right| \qquad [m \leqslant n].$$

## 5.23. Интегралы вида $\int \frac{\sin^m x}{\cos nx}\,dx$.

**1.** $$\int \frac{\sin^{2m} x}{\cos(2n+1)x}\,dx = \frac{(-1)^{n+1}}{2n+1}\left\{\ln\left|\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)\right| + \sum_{k=1}^{n}(-1)^k \cos^{2m}\frac{k\pi}{2n+1}\times\right.$$
$$\left.\times\ln\left|\operatorname{tg}\left[\frac{2n+2k+1}{4(2n+1)}\pi-\frac{x}{2}\right]\operatorname{tg}\left[\frac{2n-2k+1}{4(2n+1)}\pi-\frac{x}{2}\right]\right|\right\} \qquad [m \leqslant n].$$

**2.** $$\int \frac{\sin^{2m+1} x}{\cos(2n+1)x}\,dx = \frac{(-1)^{n+1}}{2n+1}\left[\ln|\cos x| +\right.$$
$$\left.+\sum_{k=1}^{n}(-1)^k \cos^{2m+1}\frac{k\pi}{2n+1}\ln\left|\cos^2 x - \sin^2\frac{k\pi}{2n+1}\right|\right] \qquad [m \leqslant n].$$

**3.** $$\int \frac{\sin^m x}{\cos 2nx}\,dx = \frac{1}{2n}\sum_{k=0}^{2n-1}(-1)^{n+k}\cos^m\left(\frac{2k+1}{4n}\pi\right)\times$$
$$\times\ln\left|\frac{\sin\left(\frac{2k-2n+1}{8n}\pi+\frac{x}{2}\right)}{\sin\left(\frac{2k+2n+1}{8n}\pi-\frac{x}{2}\right)}\right| \qquad [m < 2n].$$

## 5.24. Интегралы вида $\int \frac{\cos^m x}{\sin nx}\,dx$.

**1.** $$\int \frac{\cos^{2m+1} x}{\sin(2n+1)x}\,dx = \frac{1}{2n+1}\left[\ln|\sin x| +\right.$$
$$\left.+\sum_{k=1}^{n}(-1)^k \cos^{2m+1}\frac{k\pi}{2n+1}\ln\left|\sin^2 x - \sin^2\frac{k\pi}{2n+1}\right|\right] \qquad [m \leqslant n].$$

**2.** $$\int \frac{\cos^{2m} x}{\sin(2n+1)x}\,dx = \frac{1}{2n+1}\left[\ln\left|\operatorname{tg}\frac{x}{2}\right| +\right.$$
$$\left.+\sum_{k=1}^{n}(-1)^k \cos^{2m}\frac{k\pi}{2n+1}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{k\pi}{4n+2}\right)\operatorname{tg}\left(\frac{x}{2}-\frac{k\pi}{4n+2}\right)\right|\right] \qquad [m \leqslant n].$$

**3.** $$\int \frac{\cos^{2m+1} x}{\sin 2nx}\,dx = \frac{1}{2n}\left[\ln\left|\operatorname{tg}\frac{x}{2}\right| +\right.$$
$$\left.+\sum_{k=1}^{n-1}(-1)^k \cos^{2m+1}\frac{k\pi}{2n}\ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{k\pi}{4n}\right)\operatorname{tg}\left(\frac{x}{2}-\frac{k\pi}{4n}\right)\right|\right] \qquad [m < n].$$

**4.** $$\int \frac{\cos^{2m} x}{\sin 2nx}\,dx = \frac{1}{2n}\left[\ln|\sin x| + \sum_{k=1}^{n-1}(-1)^k \cos^{2m}\frac{k\pi}{2n}\ln\left|\sin^2 x - \sin^2\frac{k\pi}{2n}\right|\right] \qquad [m \leqslant n].$$

**5.25. Интегралы вида** $\int \sin^l x \cos^m x \sin^{n+1/2} 2x\,dx$.

**1.** $\displaystyle\int \sin^{2m+1} x \sin^{n+1/2} 2x\,dx = 2^{n+3/2} \int \frac{t^{2(2m+n+2)}}{(t^4+1)^{m+n+2}}\,dt \qquad [t = \sqrt{\operatorname{tg} x}\,].$

**2.** $\displaystyle = -2^{n+3/2} \int \frac{t^{2(n+1)}}{(t^4+1)^{m+n+2}}\,dt \qquad [t = \sqrt{\operatorname{ctg} x}\,].$

**3.** $\displaystyle\int \cos^{2m+1} x \sin^{n+1/2} 2x\,dx = -2^{n+3/2} \int \frac{t^{2(2m+n+2)}}{(t^4+1)^{m+n+2}}\,dt \qquad [t = \sqrt{\operatorname{ctg} x}\,].$

**4.** $\displaystyle = 2^{n+3/2} \int \frac{t^{2(n+1)}}{(t^4+1)^{m+n+2}}\,dt \qquad [t = \sqrt{\operatorname{tg} x}\,].$

**5.** $\displaystyle\int \sin x \cos^{2m} x \sin^{n+1/2} 2x\,dx = 2^{n+3/2} \int \frac{t^{2(n+2)}}{(t^4+1)^{m+n+2}}\,dt \qquad [t = \sqrt{\operatorname{tg} x}\,].$

**6.** $\displaystyle\int \cos x \sin^{2m} x \sin^{n+1/2} 2x\,dx = -2^{n+3/2} \int \frac{t^{2(n+2)}}{(t^4+1)^{m+n+2}}\,dt \qquad [t = \sqrt{\operatorname{ctg} x}\,].$

**7.** $\displaystyle\int \sin x \sqrt{\sin 2x}\,dx = -\frac{1}{2} \cos x \sqrt{\sin 2x} + \frac{1}{4}\left[\ln\left(\sin x + \cos x + \sqrt{\sin 2x}\,\right) + \arcsin(\sin x - \cos x)\right].$

**8.** $\displaystyle\int \cos x \sqrt{\sin 2x}\,dx = -\frac{1}{2} \sin x \sqrt{\sin 2x} + \frac{1}{4}\left[\ln\left(\sin x + \cos x - \sqrt{\sin 2x}\,\right) + \arcsin(\sin x - \cos x)\right].$

**5.26. Интегралы вида** $\displaystyle\int \frac{\sin^{\pm l} x \cos^{\pm m} x}{\sin^{n+1/2} 2x}\,dx$.

**1.** $\displaystyle\int \frac{\sin^{2m+1} x}{\sin^{n+1/2} 2x}\,dx = \frac{\operatorname{tg}^{3/2} x}{2^{n-1/2}} \sum_{k=0}^{n-m-1} C_{n-m-1}^{k} \frac{\operatorname{tg}^{2m-n+2k} x}{4m-2n+4k+3} \qquad [n \geqslant m+1].$

**2.** $\displaystyle\int \frac{\cos^{2m+1} x}{\sin^{n+1/2} 2x}\,dx = -\frac{\operatorname{ctg}^{3/2} x}{2^{n-1/2}} \sum_{k=0}^{n-m-1} C_{n-m-1}^{k} \frac{\operatorname{ctg}^{2m-n+2k} x}{4m-2n+4k+3} \qquad [n \geqslant m+1].$

**3.** $\displaystyle\int \frac{\sin x}{\sqrt{\sin 2x}}\,dx = \frac{1}{2}\left[\ln\left(\sin x + \cos x - \sqrt{\sin 2x}\,\right) + \arcsin(\sin x - \cos x)\right].$

**4.** $\displaystyle\int \frac{\cos x}{\sqrt{\sin 2x}}\,dx = \frac{1}{2}\left[\ln\left(\sin x + \cos x + \sqrt{\sin 2x}\,\right) + \arcsin(\sin x - \cos x)\right].$

**5.** $\displaystyle\int \frac{dx}{\sin^{2m+1} x \sin^{n+1/2} 2x} = -\frac{\operatorname{ctg}^{1/2} x}{2^{n-1/2}} \sum_{k=0}^{m+n} C_{m+n}^{k} \frac{\operatorname{ctg}^{2m+n-2k} x}{4m+2n-4k+1}.$

**6.** $\displaystyle\int \frac{dx}{\cos^{2m+1} x \sin^{n+1/2} 2x} = \frac{\operatorname{tg}^{1/2} x}{2^{n-1/2}} \sum_{k=0}^{m+n} C_{m+n}^{k} \frac{\operatorname{tg}^{2m+n-2k} x}{4m+2n-4k+1}.$

**7.** $\displaystyle\int \frac{\sin x \cos^{\pm 2m} x}{\sin^{n+1/2} 2x}\, dx = \frac{\operatorname{tg}^{3/2} x}{2^{n-1/2}} \sum_{k=0}^{n\mp m-1} C_{n\mp m-1}^{k} \frac{\operatorname{tg}^{2k-n} x}{4k-2n+3} \qquad [n \geqslant m+1].$

**8.** $\displaystyle\int \frac{\cos x \sin^{\pm 2m} x}{\sin^{n+1/2} 2x}\, dx = -\frac{\operatorname{ctg}^{3/2} x}{2^{n-1/2}} \sum_{k=0}^{n\mp m-1} C_{n\mp m-1}^{k} \frac{\operatorname{ctg}^{2k-n} x}{4k-2n+3}$

$[n \geqslant m+1].$

**9.** $\displaystyle\int \frac{\operatorname{tg}^{\pm l} x}{\sin^{2m+1} x \sin^{n+1/2} 2x}\, dx = -\frac{\operatorname{ctg}^{1/2} x}{2^{n-1/2}} \sum_{k=0}^{m+n} C_{m+n}^{k} \frac{\operatorname{ctg}^{2m+n\mp l-2k} x}{4m+2n\mp 2l-4k+1}.$

**10.** $\displaystyle\int \frac{\operatorname{tg}^{\pm l} x}{\cos^{2m+1} x \sin^{n+1/2} 2x}\, dx = \frac{\operatorname{tg}^{1/2} x}{2^{n-1/2}} \sum_{k=0}^{m+n} C_{m+n}^{k} \frac{\operatorname{tg}^{2k\pm l-n} x}{4k\pm 2l-2n+1}.$

## 5.27. Интегралы вида $\int \sin^l x \cos^m x \cos^{\pm n+1/2} 2x\, dx$.

**1.** $\displaystyle\int \sin x \cos^{n+1/2} 2x\, dx = -\frac{\cos x}{2(n+1)} \sqrt{\cos 2x} \times$

$\displaystyle\times \left[\cos^n 2x + \sum_{k=0}^{n-1} (-1)^k \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2^{k+1} n(n-1)\ldots(n-k)} \cos^{n-k-1} 2x\right] +$

$\displaystyle+ (-1)^n \frac{(2n+1)!!}{(2n+2)!!\,\sqrt{2}} \ln\left(\cos x + \sqrt{\frac{\cos 2x}{2}}\right).$

**2.** $\displaystyle\int \cos x \cos^{n+1/2} 2x\, dx = \frac{\sin x}{2(n+1)} \sqrt{\cos 2x} \times$

$\displaystyle\times \left[\cos^n 2x + \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2^{k+1} n(n-1)\ldots(n-k)} \cos^{n-k-1} 2x\right] +$

$\displaystyle+ (-1)^n \frac{(2n+1)!!}{(2n+2)!!\,\sqrt{2}} \arcsin(\sqrt{2}\sin x).$

**3.** $\displaystyle\int \sin x \sqrt{\cos 2x}\, dx = -\frac{\cos x}{2} \sqrt{\cos 2x} + \frac{1}{2\sqrt{2}} \ln\left(\cos x + \sqrt{\frac{\cos 2x}{2}}\right).$

**4.** $\displaystyle\int \cos x \sqrt{\cos 2x}\, dx = \frac{\sin x}{2} \sqrt{\cos 2x} + \frac{1}{2\sqrt{2}} \arcsin(\sqrt{2}\sin x).$

**5.** $\displaystyle\int \frac{\sin x\, dx}{\cos^{n+1/2} 2x} = -\frac{\cos x}{(2n-1)\cos^{n-1/2} 2x} \times$

$\displaystyle\times \left[1 + \sum_{k=0}^{n-2} (-1)^{k+1} \frac{2^{k+1}(n-1)(n-2)\ldots(n-k-1)}{(2n-3)(2n-5)\ldots(2n-2k-3)} \cos^{k+1} 2x\right].$

**6.** $\displaystyle\qquad = \frac{\cos x}{\sqrt{\cos 2x}} \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1} 2^k}{2k+1} C_{n-1}^{k} \frac{\cos^{2k}}{\cos^k 2x}.$

7. $$\int \frac{\cos x}{\cos^{n+1/2} 2x}\, dx = \frac{\sin x}{(2n-1)\cos^{n-1/2} 2x} \times \\ \times \left[1 + \sum_{k=0}^{n-2} \frac{2^{k+1}(n-1)(n-2)\ldots(n-k-1)}{(2n-3)(2n-5)\ldots(2n-2k-3)} \cos^{k+1} 2x\right].$$

8. $$= \frac{\sin x}{\sqrt{\cos 2x}} \sum_{k=0}^{n-1} \frac{2^k}{2k+1} C_{n-1}^k \frac{\sin^{2k} x}{\cos^k 2x}.$$

9. $$\int \frac{\sin x}{\sqrt{\cos 2x}}\, dx = \frac{1}{\sqrt{2}} \ln\left(\cos x - \sqrt{\frac{\cos 2x}{2}}\right).$$

10. $$\int \frac{\cos x}{\sqrt{\cos 2x}}\, dx = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}\,\sin x).$$

**5.28. Интегралы вида** $\int (a + b\cos x + c\sin x)^{\pm n+1/2}\, dx$.

1. $$\int (1 \pm \sin ax)^{n+1/2}\, dx = \mp\frac{1}{a}(1 \mp \sin x)^{1/2} \sum_{k=0}^{n} (-1)^k C_n^k 2^{n-k+1} \frac{(1 \mp \sin ax)^k}{2k+1}.$$

2. $$\int \frac{dx}{(1 \pm \sin ax)^{n+1/2}} = \\ = \frac{\cos ax}{2^n na(1 \pm \sin ax)^{n+1/2}} \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3)\ldots(2n-2k-1)}{(2n-2)(2n-4)\ldots(2n-2k-2)} \times \right. \\ \left. \times\, 2^{n-k-2}(1 \pm \sin ax)^{k+1}\right] \mp \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2^{n+1/2}a} \ln \frac{\sqrt{2} + (1 \mp \sin ax)^{1/2}}{\sqrt{2} - (1 \mp \sin ax)^{1/2}}.$$

3. $$\int \frac{dx}{(1 \pm \cos ax)^{n+1/2}} = \pm\frac{\sin ax}{2^n na(1 + \cos ax)^{n+1/2}} \times \\ \times \left[2^{n-1} + \sum_{k=0}^{n-2} \frac{(2n-1)(2n-3)\ldots(2n-2k-1)}{(2n-2)(2n-4)\ldots(2n-2k-2)} 2^{n-k-2}(1 \pm \cos ax)^{n+1}\right] \pm \\ \pm \frac{(2n-1)!!}{(2n)!!} \cdot \frac{1}{2^{n+1/2}a} \ln \frac{\sqrt{2} + (1 \mp \cos ax)^{1/2}}{\sqrt{2} - (1 \mp \cos ax)^{1/2}}.$$

4. $$\int \frac{(c\cos x - b\sin x)\, dx}{(a + b\cos x + c\sin x)^p} = \frac{1}{1-p}(a + b\cos x + c\sin x)^{-p+1}.$$

5. $$\int \frac{(c\cos x - b\sin x)\, dx}{a + b\cos x + c\sin x} = \ln|a + b\cos x + c\sin x|.$$

**5.29. Интегралы вида** $\int \sin^{\pm m} x(1 - k^2\sin^2 x)^{n/2}\, dx$.

Обозначение: $\Delta = \sqrt{1 - k^2\sin^2 x}$, $0 < k^2 < 1$.

1. $$\int \Delta \sin x\, dx = -\frac{\Delta\cos x}{2} - \frac{1-k^2}{2k} \ln(k\cos x + \Delta).$$

2. $$\int \Delta \sin^3 x\, dx = \\ = -\frac{2k^2\sin^2 x + 3k^2 - 1}{8k^2}\Delta\cos x + \frac{3k^4 - 2k^2 - 1}{8k^3} \ln(k\cos x + \Delta).$$

**3.** $\displaystyle\int \Delta^3 \sin x\,dx = \frac{2k^2\sin^2 x + 3k^2 - 5}{8}\Delta\cos x - \frac{3(1-k^2)^2}{8k}\ln(k\cos x + \Delta).$

**4.** $\displaystyle\int \Delta^3 \sin^3 x\,dx = \frac{8k^4\sin^4 x + 2k^2(5k^2-7)\sin^2 x + 15k^4 - 22k^2 + 3}{48k^2}\times$

$$\times\,\Delta\cos x - \frac{5k^6 - 9k^4 + 3k^2 + 1}{16k^3}\ln(k\cos x + \Delta).$$

**5.** $\displaystyle\int \frac{\Delta\,dx}{\sin x} = -\frac{1}{2}\ln\frac{\Delta + \cos x}{\Delta - \cos x} + k\ln(k\cos x + \Delta).$

**6.** $\displaystyle\int \frac{\Delta\,dx}{\sin^3 x} = -\frac{\Delta\cos x}{2\sin^2 x} + \frac{1-k^2}{4}\ln\frac{\Delta + \cos x}{\Delta - \cos x}.$

### 5.30. Интегралы вида $\displaystyle\int \frac{\sin^{\pm m} x\,dx}{(1-k^2\sin^2 x)^{n/2}}$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \frac{\sin x}{\Delta^p}\,dx = -\frac{\cos x}{(p-2)(1-k^2)\Delta^{p-2}} + \frac{p-3}{(p-2)(1-k^2)}\int \frac{\sin x}{\Delta^{p-2}}\,dx.$

**2.** $\displaystyle\int \frac{\sin x}{\Delta^{2n+1}}\,dx =$

$$= -\cos x \sum_{l=0}^{n-1} \frac{2^l(n-1)(n-2)\ldots(n-l-1)}{(2n-1)(2n-3)\ldots(2n-2l-1)(1-k^2)^{l+1}\Delta^{2n-2l-1}}.$$

**3.** $\displaystyle\qquad = -\frac{1}{(1-k^2)^n}\sum_{l=0}^{n-1} C_{n-1}^l \frac{(-k^2)^l}{2l+1}\,\frac{\cos^{2l+1} x}{\Delta^{2l+1}}.$

**4.** $\displaystyle\int \frac{\sin^p x}{\Delta^{p+3}}\,dx = -\frac{\sin^{p-3} x\cos x}{(p+1)k^2(1-k^2)\Delta^{p+1}} +$

$$+ \frac{2(p-1)k^2 - p + 2}{(p+1)k^2(1-k^2)}\int \frac{\sin^{p-2} x}{\Delta^{p+1}}\,dx + \frac{p-3}{(p+1)k^2(1-k^2)}\int \frac{\sin^{p-4} x}{\Delta^{p-1}}\,dx.$$

**5.** $\displaystyle\int \frac{\sin^n x}{\Delta}\,dx = \frac{\sin^{n-3} x}{(n-1)k^2}\Delta\cos x +$

$$+ \frac{(n-2)(1+k^2)}{(n-1)k^2}\int \frac{\sin^{n-2} x}{\Delta}1\,dx - \frac{n-3}{(n-1)k^2}\int \frac{\sin^{n-4} x}{\Delta}\,dx.$$

**6.** $\displaystyle\int \frac{\sin x}{\Delta}\,dx = \frac{1}{2k}\ln\frac{\Delta - k\cos x}{\Delta + k\cos x} = -\frac{1}{k}\ln(k\cos x + \Delta) = \frac{1}{k}\ln(\Delta - k\cos x).$

**7.** $\displaystyle\int \frac{\sin^3 x}{\Delta}\,dx = \frac{\Delta\cos x}{2k^2} - \frac{1+k^2}{2k^3}\ln(k\cos x + \Delta).$

**8.** $\displaystyle\int \frac{\sin x}{\Delta^3}\,dx = -\frac{\cos x}{(1-k^2)\Delta}.$

**9.** $\displaystyle\int \frac{\sin^3 x}{\Delta^3}\,dx = -\frac{\cos x}{k^2(1-k^2)\Delta} + \frac{1}{k^3}\ln(k\cos x + \Delta).$

**10.** $\displaystyle\int \frac{dx}{\Delta\sin^n x} =$

$$= -\frac{\Delta\cos x}{(n-1)\sin^{n-1} x} + \frac{(n-2)(1+k^2)}{n-1}\int \frac{dx}{\Delta\sin^{n-2} x} - \frac{(n-3)k^2}{n-1}\int \frac{dx}{\Delta\sin^{n-4} x}.$$

**11.** $\displaystyle\int \frac{dx}{\Delta \sin x} = -\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x} = \ln\frac{\Delta+\cos x}{\sin x}.$

**12.** $\displaystyle\int \frac{dx}{\Delta \sin^3 x} = -\frac{\Delta\cos x}{2\sin^2 x} - \frac{1+k^2}{4}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$

**5.31. Интегралы вида** $\displaystyle\int \cos^{\pm m} x(1-k^2\sin^2 x)^{n/2}\,dx.$

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \Delta\cos x\,dx = \frac{\Delta\sin x}{2} + \frac{1}{2k}\arcsin(k\sin x).$

**2.** $\displaystyle\int \Delta\cos^3 x\,dx = \frac{2k^2\cos^2 x + 2k^2 + 1}{8k^2}\Delta\sin x + \frac{4k^2-1}{8k^3}\arcsin(k\sin x).$

**3.** $\displaystyle\int \Delta^3\cos x\,dx = \frac{-2k^2\sin^2 x + 5}{8}\Delta\sin x + \frac{3}{8k}\arcsin(k\sin x).$

**4.** $\displaystyle\int \Delta^3\cos^3 x\,dx =$

$$= \frac{8k^4\sin^4 x - 2k^2(6k^2+7)\sin^2 x + 30k^2 + 3}{48k^2}\Delta\sin x + \frac{6k^2-1}{16k^3}\arcsin(k\sin x).$$

**5.** $\displaystyle\int \frac{\Delta\,dx}{\cos x} = \frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}\,\sin x}{\Delta-\sqrt{1-k^2}\,\sin x} + k\arcsin(k\sin x).$

**6.** $\displaystyle\int \frac{\Delta\,dx}{\cos^3 x} = \frac{\Delta\sin x}{2\cos^2 x} + \frac{1}{4\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}\,\sin x}{\Delta-\sqrt{1-k^2}\,\sin x}.$

**5.32. Интегралы вида** $\displaystyle\int \frac{\cos^{\pm m} x\,dx}{(1-k^2\sin^2 x)^{n/2}}.$

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \frac{\cos x}{\Delta^p}\,dx = \frac{\sin x}{(p-2)\Delta^{p-2}} + \frac{p-3}{p-2}\int\frac{\cos x}{\Delta^{p-2}}\,dx.$

**2.** $\displaystyle\int \frac{\cos x}{\Delta^{2n+1}}\,dx = -\sin x\sum_{l=0}^{n-1}\frac{2^l(n-1)(n-2)\ldots(n-l+1)}{(2n-1)(2n-3)\ldots(2n-2l-1)\Delta^{2n-2l-1}}.$

**3.** $\displaystyle\qquad = \sum_{l=0}^{n-1} C_{n-1}^l\frac{k^{2l}\sin^{2l+1} x}{(2l+1)\Delta^{2l+1}}.$

**4.** $\displaystyle\int \frac{\cos^n x}{\Delta}\,dx = \frac{\cos^{n-3} x}{(n-1)k^2}\Delta\sin x + \frac{(n-2)(2k^2-1)}{(n-1)k^2}\int\frac{\cos^{n-2} x}{\Delta}\,dx +$

$$+ \frac{(n-3)(1-k^2)}{(n-1)k^2}\int\frac{\cos^{n-4} x}{\Delta}\,dx.$$

**5.** $\displaystyle\int \frac{\cos x}{\Delta}\,dx = \frac{1}{k}\arcsin(k\sin x) = \frac{1}{k}\operatorname{arctg}\frac{k\sin x}{\Delta}.$

**6.** $\displaystyle\int \frac{\cos^3 x}{\Delta}\,dx = \frac{\Delta\sin x}{2k^2} + \frac{2k^2-1}{2k^3}\arcsin(k\sin x).$

**7.** $\displaystyle\int \frac{\cos x}{\Delta^3}\,dx = \frac{\sin x}{\Delta}.$

**8.** $\displaystyle\int \frac{\cos^3 x}{\Delta^3}\,dx = -\frac{(1-k^2)\sin x}{k^2\Delta} + \frac{1}{k^3}\arcsin(k\sin x).$

**9.** $\displaystyle\int \frac{dx}{\Delta\cos^n x} = \frac{\Delta\sin x}{(n-1)(1-k^2)\cos^{n-1} x} - {}$

$$-\frac{(n-2)(2k^2-1)}{(n-1)(1-k^2)}\int\frac{dx}{\Delta\cos^{n-2}x} + \frac{(n-3)k^2}{(n-1)(1-k^2)}\int\frac{dx}{\Delta\cos^{n-4}x}.$$

**10.** $\displaystyle\int \frac{dx}{\Delta\cos x} = -\frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta-\sqrt{1-k^2}\,\sin x}{\Delta+\sqrt{1-k^2}\,\sin x}.$

**11.** $\displaystyle\int \frac{dx}{\Delta\cos^3 x} = \frac{\Delta\sin x}{2(1-k^2)\cos^2 x} + \frac{2k^2-1}{4(1-k^2)^{3/2}}\ln\frac{\Delta-\sqrt{1-k^2}\,\sin x}{\Delta+\sqrt{1-k^2}\,\sin x}.$

### **5.33.** Интегралы вида $\displaystyle\int \sin^m x\cos^n x(1-k^2\sin^2 x)^p\,dx$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \Delta^p\sin^m x\cos^n x\,dx = \frac{1}{(m+n+p)k^2}\Big\{\Delta^{p+2}\sin^{m-3}x\cos^{n+1}x + {}$

$$+[m+n-2+(m+p-1)k^2]\int\Delta^p\sin^{m-2}x\cos^n x\,dx - {}$$

$$-(m-3)\int\Delta^p\sin^{m-4}x\cos^n x\,dx\Big\}.$$

**2.** $\displaystyle\qquad = \frac{1}{(m+n+p)k^2}\Big\{\Delta^{p+2}\sin^{m+1}x\cos^{n-3}x + {}$

$$+[(n+p-1)k^2-(m+n-2)(1-k^2)]\int\Delta^p\sin^m x\cos^{n-2}x\,dx + {}$$

$$+(n-3)(1-k^2)\int\Delta^p\sin^m x\cos^{n-4}x\,dx\Big\}.$$

**3.** $\displaystyle\int \Delta^p\sin x\cos^n x\,dx = {}$

$$= -\frac{\Delta^{p+2}\cos^{n-1}x}{(n+p+1)k^2} - \frac{(n-1)(1-k^2)}{(n+p+1)k^2}\int\Delta^p\sin x\cos^{n-2}x\,dx.$$

**4.** $\displaystyle\int \Delta^p\sin^m x\cos x\,dx = {}$

$$= -\frac{\Delta^{p+2}\sin^{m-1}x}{(m+p+1)k^2} + \frac{m-1}{(m+p+1)k^2}\int\Delta^p\sin^{m-2}x\cos x\,dx.$$

**5.** $\displaystyle\int \Delta\sin x\cos x\,dx = -\frac{\Delta^3}{3k^2}.$

**6.** $\displaystyle\int \Delta\sin x\cos^2 x\,dx = {}$

$$= -\frac{2k^2\cos^2 x+1-k^2}{8k^2}\Delta\cos x + \frac{(1-k^2)^2}{8k^3}\ln(k\cos x+\Delta).$$

**7.** $\displaystyle\int \Delta\sin x\cos^3 x\,dx = -\frac{3k^4\cos^4 x - k^2(5k^2+1)\sin^2 x + 5k^2 - 2}{15k^4}\Delta.$

**8.** $\displaystyle\int \Delta\sin x\cos^4 x\,dx = \frac{-8k^4\sin^4 x + 2k^2(7k^2+1)\sin^2 x - 3k^4 - 8k^2 + 3}{48k^4}\times$

$$\times\Delta\cos x - \frac{1-k^2}{16k^5}\ln(k\cos x+\Delta).$$

**9.** $\int \Delta \sin^2 x \cos x\, dx = \dfrac{2k^2 \sin^2 x - 1}{8k^2} \Delta \sin x + \dfrac{1}{8k^3} \arcsin (k \sin x).$

**10.** $\int \Delta \sin^2 x \cos^3 x\, dx = \dfrac{-8k^4 \sin^4 x + 2k^2(6k^2+1)\sin^2 x - 6k^2 + 3}{48k^4} \times$
$$\times \Delta \sin x + \frac{2k^2-1}{16k^5} \arcsin (k \sin x).$$

**11.** $\int \Delta \sin^3 x \cos x\, dx = \dfrac{3k^4 \sin^4 x - k^2 \sin^2 x - 2}{15k^4} \Delta.$

**12.** $\int \Delta \sin^3 x \cos^2 x\, dx = \dfrac{8k^4 \sin^4 x - 2k^2(k^2+1)\sin^2 x - 3k^4 + 2k^2 - 3}{48k^4} \times$
$$\times \Delta \cos x + \frac{(1-k^2)^2(k^2+1)}{16k^5} \ln (k \cos x + \Delta).$$

**13.** $\int \Delta \sin^4 x \cos x\, dx =$
$$= \frac{8k^4 \sin^4 x - 2k^2 \sin^2 x - 3}{48k^4} \Delta \sin x + \frac{1}{16k^5} \arcsin (k \sin x).$$

**14.** $\int \Delta^3 \sin x \cos x\, dx = -\dfrac{\Delta^5}{5k^2}.$

**15.** $\int \Delta^3 \sin x \cos^2 x\, dx = \dfrac{-8k^4 \sin^4 x + 2k^2(k^2+7)\sin^2 x + 3k^4 - 8k^2 - 3}{48k^2} \times$
$$\times \Delta \cos x + \frac{(1-k^2)^3}{16k^3} \ln (k \cos x + \Delta).$$

**16.** $\int \Delta^3 \sin^2 x \cos x\, dx =$
$$= \frac{-8k^4 \sin^4 x + 14k^2 \sin^2 x - 3}{48k^2} \Delta \sin x + (1/(16k^3)) \arcsin (k \sin x).$$

### 5.34. Интегралы вида $\int \dfrac{\sin^m x}{\cos^n x} (1 - k^2 \sin^2 x)^{1/2}\, dx$.

Обозначение: $\Delta = \sqrt{1 - k^2 \sin^2 x}$, $0 < k^2 < 1$.

**1.** $\int \dfrac{\sin x}{\cos x} \Delta\, dx = \int \Delta \operatorname{tg} x\, dx = -\Delta + \dfrac{\sqrt{1-k^2}}{2} \ln \dfrac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$

**2.** $\int \dfrac{\sin x}{\cos^2 x} \Delta\, dx = \dfrac{\Delta}{\cos x} - k \ln (k \cos x + \Delta).$

**3.** $\int \dfrac{\sin x}{\cos^3 x} \Delta\, dx = \dfrac{\Delta}{2\cos^2 x} + \dfrac{k^2}{4\sqrt{1-k^2}} \ln \dfrac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$

**4.** $\int \dfrac{\sin x}{\cos^4 x} \Delta\, dx = \dfrac{-(2k^2+1)k^2 \sin^2 x + 3k^4 - k^2 + 1}{3(1-k^2)\cos^3 x} \Delta.$

**5.** $\int \dfrac{\sin^2 x}{\cos x} \Delta\, dx = -\dfrac{\Delta \sin x}{2} + \dfrac{2k^2-1}{2k} \arcsin (k \sin x) +$
$$+ \frac{\sqrt{1-k^2}}{2} \ln \frac{\Delta + \sqrt{1-k^2}\, \sin x}{\Delta - \sqrt{1-k^2}\, \sin x}.$$

**6.** $\int \dfrac{\sin^2 x}{\cos^3 x} \Delta\, dx =$
$$= \frac{\sin x}{2\cos^2 x} \Delta + \frac{2k^2-1}{4\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}\, \sin x}{\Delta - \sqrt{1-k^2}\, \sin x} - k \arcsin (k \sin x).$$

**7.** $\displaystyle\int \frac{\sin^3 x}{\cos x}\Delta\,dx = -\frac{k^2\sin x + 3k^2 - 1}{3k^2}\Delta + \frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$

**8.** $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\Delta\,dx = -\frac{\sin^2 x - 3}{2\cos x}\Delta - \frac{3k^2-1}{2k}\ln(k\cos x + \Delta).$

**9.** $\displaystyle\int \frac{\sin^4 x}{\cos x}\Delta\,dx = -\frac{2k^2\sin^2 x + 4k^2 - 1}{8k^2}\Delta\sin x +$

$$+\frac{8k^4-4k^2-1}{8k^3}\arcsin(k\sin x) + \frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}\,\sin x}{\Delta-\sqrt{1-k^2}\,\sin x}.$$

## 5.35. Интегралы вида $\displaystyle\int \frac{\cos^m x}{\sin^n x}(1-k^2\sin^2 x)^{1/2}\,dx$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \frac{\cos x}{\sin x}\Delta\,dx = \int \Delta\operatorname{ctg} x\,dx - \Delta + \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}.$

**2.** $\displaystyle\int \frac{\cos^2 x}{\sin x}\Delta\,dx = \frac{\Delta\cos x}{2} + \frac{k^2+1}{2k}\ln(k\cos x+\Delta) + \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$

**3.** $\displaystyle\int \frac{\cos^3 x}{\sin x}\Delta\,dx = -\frac{k^2\sin^2 x - 3k^2 - 1}{3k^2}\Delta + \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta}.$

**4.** $\displaystyle\int \frac{\cos^4 x}{\sin x}\Delta\,dx = \frac{-2k^2\sin^2 x + 5k^2 + 1}{8k^2}\Delta\cos x +$

$$+\frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x} + \frac{3k^4+6k^2-1}{8k^3}\ln(k\cos x+\Delta).$$

**5.** $\displaystyle\int \frac{\cos x}{\sin^2 x}\Delta\,dx = -\frac{\Delta}{\sin x} - k\arcsin(k\sin x).$

**6.** $\displaystyle\int \frac{\cos^3 x}{\sin^2 x}\Delta\,dx = -\frac{\sin^2 x + 2}{2\sin x}\Delta - \frac{2k^2+1}{2k}\arcsin(k\sin x).$

**7.** $\displaystyle\int \frac{\cos x}{\sin^3 x}\Delta\,dx = -\frac{\Delta}{2\sin^2 x} + \frac{k^2}{4}\ln\frac{1+\Delta}{1-\Delta}.$

**8.** $\displaystyle\int \frac{\cos x}{\sin^4 x}\Delta\,dx = -\frac{\Delta^3}{3\sin^3 x}.$

**9.** $\displaystyle\int \frac{\cos^2 x}{\sin^3 x}\Delta\,dx = -\frac{\cos x}{2\sin^2 x}\Delta - \frac{k^2+1}{4}\ln\frac{\Delta+\cos x}{\Delta-\cos x} - k\ln(k\cos x+\Delta).$

## 5.36. Интегралы вида $\displaystyle\int \frac{\sin^p x\cos^q x}{(1-k^2\sin^2 x)^r}\,dx$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \frac{\sin^p x\cos^q x}{\Delta^r}\,dx = \frac{-k^2\sin^{p+1} x\cos^{q+1} x}{(r-2)(1-k^2)\Delta^{r-2}} + \frac{p-q-(2-k^2)(p-r+3)}{(r-2)(1-k^2)}\times$

$$\times\int \frac{\sin^p x\cos^q x}{\Delta^{r-2}}\,dx + \frac{p+q-r+4}{(r-2)(1-k^2)}\int \frac{\sin^p x\cos^q x}{\Delta^{r-4}}\,dx.$$

**2.** $\displaystyle\qquad = \frac{1}{k^2}\int \frac{\sin^{p-2} x\cos^q x}{\Delta^r}\,dx - \frac{1}{k^2}\int \frac{\sin^{p-2} x\cos^q x}{\Delta^{r-2}}\,dx.$

**3.** $\displaystyle\qquad = \frac{k^2-1}{k^2}\int \frac{\sin^p x\cos^{q-2} x}{\Delta^r}\,dx + \frac{1}{k^2}\int \frac{\sin^p x\cos^{q-2} x}{\Delta^{r-2}}\,dx.$

**4.** $\displaystyle\int \frac{\sin x\cos x}{\Delta^r}\,dx = \frac{1}{(r-2)k^2\Delta^{r-2}}.$

**5.** $\displaystyle\int \frac{\sin x\cos x}{\Delta}\,dx = -\frac{\Delta}{k^2}.$

**6.** $\displaystyle\int \frac{\sin x\cos^2 x}{\Delta}\,dx = -\frac{\Delta\cos x}{2k^2} + \frac{1-k^2}{2k^3}\ln(k\cos x+\Delta).$

**7.** $\displaystyle\int \frac{\sin x\cos^3 x}{\Delta}\,dx = -\frac{1}{3k^4}\,(k^2\cos^2 x - 2 + 2k^2)\Delta.$

**8.** $\displaystyle\int \frac{\sin x\cos^4 x}{\Delta}\,dx =$

$$= \frac{3-5k^2+2k^2\sin^2 x}{8k^4}\Delta\cos x - \frac{3k^4-6k^2+3}{8k^5}\ln(k\cos x+\Delta).$$

**9.** $\displaystyle\int \frac{\sin^2 x\cos x}{\Delta}\,dx = -\frac{\Delta\sin x}{2k^2} + \frac{\arcsin(k\sin x)}{2k^3}.$

**10.** $\displaystyle\int \frac{\sin^2 x\cos^3 x}{\Delta}\,dx =$

$$= -\frac{2k^2\cos^2 x + 2k^2 - 3}{8k^4}\Delta\sin x + \frac{4k^2-3}{8k^5}\arcsin(k\sin x).$$

**11.** $\displaystyle\int \frac{\sin^3 x\cos x}{\Delta}\,dx = -\frac{1}{3k^4}\,(2+k^2\sin^2 x)\Delta.$

**12.** $\displaystyle\int \frac{\sin^3 x\cos^2 x}{\Delta}\,dx =$

$$= \frac{2k^2\cos^2 x - k^2 - 3}{8k^4}\Delta\cos x - \frac{k^4+2k^2-3}{8k^5}\ln(k\cos x+\Delta).$$

**13.** $\displaystyle\int \frac{\sin^3 x\cos^3 x}{\Delta}\,dx = \frac{3k^4\sin^4 x - (5k^4-4k^2)\sin^2 x - 10k^2 + 8}{15k^6}\Delta.$

**14.** $\displaystyle\int \frac{\sin^3 x\cos^4 x}{\Delta}\,dx = \frac{8k^4\sin^4 x - 2k^2(7k^2-5)\sin^2 x + 3k^4 - 22k^2 + 15}{48k^6}\times$

$$\times\,\Delta\cos x - \frac{k^6+3k^4-9k^2+5}{16k^7}\ln(k\cos x+\Delta).$$

**15.** $\displaystyle\int \frac{\sin^4 x\cos x}{\Delta}\,dx = -\frac{2k^2\sin^2 x+3}{8k^4}\Delta\sin x + \frac{3}{8k^5}\arcsin(k\sin x).$

**16.** $\displaystyle\int \frac{\sin^4 x\cos^3 x}{\Delta}\,dx = \frac{8k^4\sin^4 x - 2k^2(6k^2-5)\sin^2 x - 18k^2 + 15}{48k^6}\Delta\sin x +$

$$+ \frac{6k^2-5}{16k^7}\arcsin(k\sin x).$$

**17.** $\displaystyle\int \frac{\sin^m x\cos^n x}{\Delta^3}\,dx = \frac{\sin^{m-1} x\cos^{n-1} x}{k^2\Delta} -$

$$- \frac{m-1}{k^2}\int \frac{\sin^{m-2} x\cos^n x}{\Delta}\,dx + \frac{n-1}{k^2}\int \frac{\sin^m x\cos^{n-2} x}{\Delta}\,dx.$$

**18.** $\displaystyle\int \frac{\sin x\cos x}{\Delta^3}\,dx = \frac{1}{k^2\Delta}.$

**19.** $\displaystyle\int \frac{\sin x\cos^2 x}{\Delta^3}\,dx = \frac{\cos x}{k^2\Delta} - \frac{1}{k^3}\ln(k\cos x + \Delta).$

**20.** $\displaystyle\int \frac{\sin x\cos^3 x}{\Delta^3}\,dx = \frac{k^2\sin^2 x + k^2 - 2}{k^4\Delta}.$

**21.** $\displaystyle\int \frac{\sin x\cos^4 x}{\Delta^3}\,dx = \frac{k^2\sin^2 x + 2k^2 - 3}{2k^4\Delta}\cos x + \frac{3(1-k^2)}{2k^5}\ln(k\cos x + \Delta).$

**22.** $\displaystyle\int \frac{\sin^2 x\cos x}{\Delta^3}\,dx = \frac{\sin x}{k^2\Delta} - \frac{1}{k^3}\arcsin(k\sin x).$

**23.** $\displaystyle\int \frac{\sin^2 x\cos^3 x}{\Delta^3}\,dx = \frac{k^2\sin^2 x + 2k^2 - 3}{2k^4\Delta}\sin x - \frac{2k^2-3}{2k^5}\arcsin(k\sin x).$

**24.** $\displaystyle\int \frac{\sin^3 x\cos x}{\Delta^3}\,dx = \frac{2 - k^2\sin^2 x}{k^4\Delta}.$

**25.** $\displaystyle\int \frac{\sin^3 x\cos^2 x}{\Delta^3}\,dx = \frac{3 - k^2\sin^2 x}{2k^4\Delta}\cos x + \frac{k^2-3}{2k^5}\ln(k\cos x + \Delta).$

**26.** $\displaystyle\int \frac{\sin^4 x\cos x}{\Delta^3}\,dx = \frac{3 - k^2\sin^2 x}{2k^4\Delta}\sin x - \frac{3}{2k^5}\arcsin(k\sin x).$

## 5.37. Интегралы вида $\displaystyle\int \frac{\sin^p x}{\cos^q x}\,\frac{dx}{(1-k^2\sin^2 x)^r}$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $$\int \frac{\sin^p x}{\cos^q x}\,\frac{dx}{\Delta^r} = \frac{\sin^{p+1} x}{(q-1)(1-k^2)\Delta^{r-2}\cos^{q-1} x} - \frac{p-q+2-(p-2q-r+5)k^2}{(q-1)(1-k^2)}\int \frac{\sin^p x}{\cos^{q-2} x}\,\frac{dx}{\Delta^r} - \frac{(p-q-r+4)k^2}{(q-1)(1-k^2)}\int \frac{\sin^p x}{\cos^{q-4} x}\,\frac{dx}{\Delta^r}.$$

**2.** $$\int \frac{\sin x}{\cos^{2m} x}\,\frac{dx}{\Delta^{2n+1}} = (-1)^{m+n}\frac{k^{2(m+n-1)}}{(1-k^2)^{m+n}}\sum_{l=0}^{m+n-1} C_{m+n-1}^{l}\frac{(-1)^l}{(2n-2l-1)k^{2l}}\left(\frac{\cos x}{\Delta}\right)^{2n-2l-1}.$$

**3.** $\displaystyle\int \frac{\sin x}{\cos x}\,\frac{dx}{\Delta} = \int \operatorname{tg} x\,\frac{dx}{\Delta} = \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$

**4.** $\displaystyle\int \frac{\sin x}{\cos^2 x}\,\frac{dx}{\Delta} = \frac{\Delta}{(1-k^2)\cos x}.$

**5.** $$\int \frac{\sin x}{\cos^3 x}\,\frac{dx}{\Delta} = \int \operatorname{tg} x\,(1+\operatorname{tg}^2 x)\frac{dx}{\Delta} = \frac{\Delta}{2(1-k^2)\cos^2 x} - \frac{k^2}{4(1-k^2)^{3/2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

**6.** $\displaystyle\int \frac{\sin x}{\cos^4 x}\,\frac{dx}{\Delta} = -\frac{2k^2\cos^2 x - 1 + k^2}{3(1-k^2)^2\cos^3 x}\Delta.$

**7.** $\displaystyle\int \frac{\sin^2 x}{\cos x}\,\frac{dx}{\Delta} = \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}\,\sin x}{\Delta-\sqrt{1-k^2}\,\sin x} - \frac{1}{k}\arcsin(k\sin x).$

**8.** $\displaystyle\int \frac{\sin^2 x}{\cos^3 x}\,\frac{dx}{\Delta} = \frac{\Delta \sin x}{2(1-k^2)\cos^2 x} - \frac{1}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2}\,\sin x}{\Delta - \sqrt{1-k^2}\,\sin x}.$

**9.** $\displaystyle\int \frac{\sin^3 x}{\cos x}\,\frac{dx}{\Delta} = \frac{\Delta}{k^2} + \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}}.$

**10.** $\displaystyle\int \frac{\sin^4 x}{\cos x}\,\frac{dx}{\Delta} = \frac{\Delta \sin x}{2k^2} +$

$$+ \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}\,\sin x}{\Delta - \sqrt{1-k^2}\,\sin x} - \frac{2k^2+1}{2k^3} \arcsin(k \sin x).$$

**11.** $\displaystyle\int \frac{\sin^3 x}{\cos^2 x}\,\frac{dx}{\Delta} = \frac{\Delta}{(1-k^2)\cos x} + \frac{1}{k} \ln(k\cos x + \Delta).$

**12.** $\displaystyle\int \operatorname{tg}^n x\,\frac{dx}{\Delta} = \frac{\operatorname{tg}^{n-3} x}{(n-1)(1-k^2)}\,\frac{\Delta}{\cos^2 x} -$

$$- \frac{(n-2)(2-k^2)}{(n-1)(1-k^2)} \int \frac{\operatorname{tg}^{n-2} x}{\Delta}\,dx - \frac{n-3}{(n-1)(1-k^2)} \int \frac{\operatorname{tg}^{n-4} x}{\Delta}\,dx.$$

**5.38. Интегралы вида** $\displaystyle\int \frac{\cos^p x}{\sin^q x}\,\frac{dx}{(1-k^2\sin^2 x)^r}$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \frac{\cos^q x}{\sin^p x}\,\frac{dx}{\Delta^r} = -\frac{\cos^{q+1} x}{(p-1)\sin^{p-1} x\,\Delta^{r-2}} + \frac{p-q-2+(p+r-3)k^2}{p-1} \times$

$$\times \int \frac{\cos^q x}{\sin^{p-2} x}\,\frac{dx}{\Delta^r} + \frac{(p-q+r-4)k^2}{p-1} \int \frac{\cos^q x}{\sin^{p-4} x}\,\frac{dx}{\Delta^r}.$$

**2.** $\displaystyle\int \frac{\cos x}{\sin^{2m} x}\,\frac{dx}{\Delta^{2n+1}} = \sum_{l=0}^{m+n-1} C_{m+n-1}^{l}\,\frac{k^{2(m+n-l-1)}}{2n-2l-1} \left(\frac{\sin x}{\Delta}\right)^{2n-2l-1}.$

**3.** $\displaystyle\int \frac{\cos x}{\sin x}\,\frac{dx}{\Delta} = \int \operatorname{ctg} x\,\frac{dx}{\Delta} = \frac{1}{2} \ln \frac{1-\Delta}{1+\Delta}.$

**4.** $\displaystyle\int \frac{\cos^2 x}{\sin x}\,\frac{dx}{\Delta} = \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{1}{k} \ln(k\cos x + \Delta).$

**5.** $\displaystyle\int \frac{\cos^3 x}{\sin x}\,\frac{dx}{\Delta} = \frac{\Delta}{k^2} - \frac{1}{2} \ln \frac{1+\Delta}{1-\Delta}.$

**6.** $\displaystyle\int \frac{\cos^4 x}{\sin x}\,\frac{dx}{\Delta} = \frac{\Delta \cos x}{2k^2} + \frac{1}{2} \ln \frac{\Delta + \cos x}{\Delta - \cos x} + \frac{3k^2-1}{2k^3} \ln(k\cos x + \Delta).$

**7.** $\displaystyle\int \frac{\cos x}{\sin^2 x}\,\frac{dx}{\Delta} = -\frac{\Delta}{\sin x}.$

**8.** $\displaystyle\int \frac{\cos^3 x}{\sin^2 x}\,\frac{dx}{\Delta} = -\frac{\Delta}{\sin x} - \frac{1}{k} \arcsin(k \sin x).$

**9.** $\displaystyle\int \frac{\cos x}{\sin^3 x}\,\frac{dx}{\Delta} = -\frac{\Delta}{2\sin^2 x} - \frac{k^2}{4} \ln \frac{1+\Delta}{1-\Delta}.$

**10.** $\displaystyle\int \frac{\cos^2 x}{\sin^3 x}\,\frac{dx}{\Delta} = -\frac{\Delta \cos x}{2\sin^2 x} + \frac{1-k^2}{4} \ln \frac{\Delta + \cos x}{\Delta - \cos x}.$

**11.** $\displaystyle\int \frac{\cos x}{\sin^4 x}\,\frac{dx}{\Delta} = -\frac{2k^2\sin^2 x + 1}{3\sin^3 x}\Delta.$

**12.** $$\int \operatorname{ctg}^n x\,\frac{dx}{\Delta} = -\frac{\operatorname{ctg}^{n-1} x}{n-1}\,\frac{\Delta}{\cos^2 x} - \frac{n-2}{n-1}(2-k^2)\times \int \frac{\operatorname{ctg}^{n-2} x}{\Delta}\,dx - \frac{n-3}{n-1}(1-k^2)\int \frac{\operatorname{ctg}^{n-4} x}{\Delta}\,dx.$$

### 5.39. Интегралы вида $\int \frac{(1-k^2\sin^2 x)^{1/2}}{\sin^m x\cos^n x}\,dx$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $$\int \frac{\Delta\,dx}{\sin x\cos x} = \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta} + \frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

**2.** $$\int \frac{\Delta\,dx}{\sin x\cos^2 x} = \frac{\Delta}{\cos x} + \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

**3.** $$\int \frac{\Delta\,dx}{\sin x\cos^3 x} = \frac{\Delta}{2\cos^2 x} - \frac{1}{2}\ln\frac{1+\Delta}{1-\Delta} + \frac{2-k^2}{4\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

**4.** $$\int \frac{\Delta\,dx}{\sin x\cos^4 x} = \frac{(2k^2-3)\sin^2 x - 3k^2 + 4}{3(1-k^2)\cos^3 x}\Delta + \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

**5.** $$\int \frac{\Delta\,dx}{\sin^2 x\cos x} = -\frac{\Delta}{\sin x} - \frac{1+k^2}{2\sqrt{1-k^2}}\ln\frac{\Delta-\sqrt{1-k^2}\,\sin x}{\Delta+\sqrt{1-k^2}\,\sin x}.$$

**6.** $$\int \frac{\Delta\,dx}{\sin^2 x\cos^3 x} = \frac{3\sin^2 x - 2}{2\sin x\cos^2 x}\Delta - \frac{2k^2-3}{4\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}\,\sin x}{\Delta-\sqrt{1-k^2}\,\sin x}.$$

**7.** $$\int \frac{\Delta\,dx}{\sin^3 x\cos x} = -\frac{\Delta}{2\sin^2 x} + \frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}} + \frac{k^2-2}{4}\ln\frac{1+\Delta}{1-\Delta}.$$

**8.** $$\int \frac{\Delta\,dx}{\sin^3 x\cos^2 x} = \frac{3\sin^2 x - 1}{2\sin^2 x\cos x}\Delta + \frac{k^2-3}{4}\ln\frac{\Delta-\cos x}{\Delta+\cos x}.$$

**9.** $$\int \frac{\Delta\,dx}{\sin^4 x\cos x} = -\frac{(3-k^2)\sin^2 x + 1}{3\sin^3 x}\Delta - \frac{\sqrt{1-k^2}}{2}\ln\frac{\Delta-\sqrt{1-k^2}\,\sin x}{\Delta+\sqrt{1-k^2}\,\sin x}.$$

### 5.40. Интегралы вида $\int \frac{dx}{\sin^m x\cos^n x(1-k^2\sin^2 x)^{1/2}}$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $$\int \frac{dx}{\Delta\sin x\cos x} = \int(\operatorname{tg} x + \operatorname{ctg} x)\frac{dx}{\Delta} = \frac{1}{2}\ln\frac{1-\Delta}{1+\Delta} + \frac{1}{2\sqrt{1-k^2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

**2.** $$\int \frac{dx}{\Delta\sin x\cos^2 x} = \frac{\Delta}{(1-k^2)\cos x} + \frac{1}{2}\ln\frac{\Delta-\cos x}{\Delta+\cos x}.$$

**3.** $$\int \frac{dx}{\Delta\sin x\cos^3 x} = \int(\operatorname{ctg} x + 2\operatorname{tg} x + \operatorname{tg}^3 x)\frac{dx}{\Delta} = \frac{\Delta}{2(1-k^2)\cos^2 x} - \frac{1}{2}\ln\frac{1+\Delta}{1-\Delta} + \frac{2-3k^2}{4(1-k^2)^{3/2}}\ln\frac{\Delta+\sqrt{1-k^2}}{\Delta-\sqrt{1-k^2}}.$$

**4.** $$\int \frac{dx}{\Delta\sin x\cos^4 x} = \frac{(5k^2-3)\sin^2 x - 6k^2 + 4}{3(1-k^2)^2\cos^3 x}\Delta - \frac{1}{2}\ln\frac{\Delta+\cos x}{\Delta-\cos x}.$$

**5.** $\displaystyle\int \frac{dx}{\Delta \sin^2 x \cos x} = -\frac{\Delta}{\sin x} - \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta - \sqrt{1-k^2}\,\sin x}{\Delta + \sqrt{1-k^2}\,\sin x}.$

**6.** $\displaystyle\int \frac{dx}{\Delta \sin^2 x \cos^3 x} = \frac{(3-2k^2)\sin^2 x - 2(1-k^2)}{2(1-k^2)\sin x \cos^2 x}\Delta - {}$

$$- \frac{4k^2-3}{4(1-k^2)^{3/2}} \ln \frac{\Delta + \sqrt{1-k^2}\,\sin x}{\Delta - \sqrt{1-k^2}\,\sin x}.$$

**7.** $\displaystyle\int \frac{dx}{\Delta \sin^3 x \cos x} = -\frac{\Delta}{2\sin^2 x} + \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta + \sqrt{1-k^2}}{\Delta - \sqrt{1-k^2}} - \frac{k^2+2}{4} \ln \frac{1+\Delta}{1-\Delta}.$

**8.** $\displaystyle\int \frac{dx}{\Delta \sin^3 x \cos^2 x} = \frac{(3-k^2)\sin^2 x - 1 + k^2}{2(1-k^2)\sin^2 x \cos x}\Delta + \frac{k^2+3}{4} \ln \frac{\Delta - \cos x}{\Delta + \cos x}.$

**9.** $\displaystyle\int \frac{dx}{\Delta \sin^4 x \cos x} = -\frac{(3+2k^2)\sin^2 x + 1}{3\sin^3 x}\Delta - \frac{1}{2\sqrt{1-k^2}} \ln \frac{\Delta - \sqrt{1-k^2}\,\sin x}{\Delta + \sqrt{1-k^2}\,\sin x}.$

### 5.41. Интегралы вида $\displaystyle\int \frac{\sin^m x \cos^n x\, dx}{(1+a\sin^2 x)(1-k^2\sin^2 x)^{1/2}}$.

Обозначение: $\Delta = \sqrt{1-k^2\sin^2 x}$, $0 < k^2 < 1$.

**1.** $\displaystyle\int \frac{\sin x\, dx}{(1+a\sin^2 x)\Delta} = \frac{1}{\sqrt{(1+a)(k^2+a)}} \times$

$$\times \ln \frac{\sqrt{1+a}\,\Delta - \sqrt{k^2+a}\,\cos x}{\left(\sqrt{1+a} - \sqrt{k^2+a}\right)\sqrt{1+a\sin^2 x}} \qquad [a > -k^2].$$

**2.** $\displaystyle\qquad = \frac{1}{2\sqrt{-(1+a)(k^2+a)}} \arcsin \frac{(1+a)\Delta^2 + (k^2+a)\cos^2 x}{(1-k^2)(1+a\sin^2 x)}$

$$[-1 < a < -k^2].$$

**3.** $\displaystyle\qquad = -\frac{1}{\sqrt{(1+a)(k^2+a)}} \times$

$$\times \ln \frac{\sqrt{-(k^2+a)}\,\cos x - \sqrt{-(1+a)}\,\Delta}{\left[\sqrt{-(k^2+a)} - \sqrt{-(1+a)}\,\right]\sqrt{1+a\sin^2 x}}$$

$$[a < -1;\ 0 \leqslant \sin^2 x < -1/a].$$

**4.** $\displaystyle\int \frac{\sin x\, dx}{(1-\sin^2 x)\Delta} = \frac{\Delta}{(1-k^2)\cos x}.$

**5.** $\displaystyle\int \frac{\sin x\, dx}{(1-k^2\sin^2 x)\Delta} = -\frac{\cos x}{(1-k^2)\Delta}.$

**6.** $\displaystyle\int \frac{\cos x\, dx}{(1+a\sin^2 x)\Delta} = \frac{1}{\sqrt{k^2+a}} \operatorname{arctg} \frac{\sqrt{k^2+a}\,\sin x}{\Delta} \qquad [a > k^2].$

**7.** $\displaystyle\qquad = \frac{1}{2\sqrt{-(k^2+a)}} \ln \frac{\Delta + \sqrt{-(k^2+a)}\,\sin x}{\Delta - \sqrt{-(k^2+a)}\,\sin x}$

$$[a < -k^2;\ 0 \leqslant \sin^2 x < -1/a].$$

**8.** $\displaystyle\int \frac{\cos x\, dx}{(1-k^2\sin^2 x)\Delta} = \frac{\sin x}{\Delta}.$

**5.42. Интегралы вида** $\displaystyle\int \frac{\sin^{\pm m} x \cos^{\pm n} x}{\sqrt{1+a^2\sin^2 x}}\,dx.$

**1.** $$\int \frac{\sin x\,dx}{\sqrt{1+a^2\sin^2 x}} = -\frac{1}{a}\arcsin\left(\frac{a\cos x}{\sqrt{1+a^2}}\right).$$

**2.** $$\int \frac{\cos x\,dx}{\sqrt{1+a^2\sin^2 x}} = \frac{1}{a}\ln\left(a\sin x + \sqrt{1+a^2\sin^2 x}\,\right).$$

**3.** $$\int \frac{dx}{\sin x\sqrt{1+a^2\sin^2 x}} = \frac{1}{2}\ln\frac{\sqrt{1+a^2\sin^2 x} - \cos x}{\sqrt{1+a^2\sin^2 x} + \cos x}.$$

**4.** $$\int \frac{dx}{\cos x\sqrt{1+a^2\sin^2 x}} = \frac{1}{2\sqrt{1+a^2}}\ln\frac{\sqrt{1+a^2\sin^2 x} + \sqrt{1+a^2}\,\sin x}{\sqrt{1+a^2\sin^2 x} - \sqrt{1+a^2}\,\sin x}.$$

**5.** $$\int \frac{\operatorname{tg} x\,dx}{\sqrt{1+a^2\sin^2 x}} = \frac{1}{2\sqrt{1+a^2}}\ln\frac{\sqrt{1+a^2\sin^2 x} + \sqrt{1+a^2}}{\sqrt{1+a^2\sin^2 x} - \sqrt{1+a^2}}.$$

**6.** $$\int \frac{\operatorname{ctg} x\,dx}{\sqrt{1+a^2\sin^2 x}} = \frac{1}{2}\ln\frac{1 - \sqrt{1+a^2\sin^2 x}}{1 + \sqrt{1+a^2\sin^2 x}}.$$

**5.43. Интегралы вида** $\displaystyle\int \frac{\sin^{\pm m} x \cos^{\pm n} x}{\sqrt{a^2\sin^2 x - 1}}\,dx.$

Обозначение: $a > 1$.

**1.** $$\int \frac{\sin x\,dx}{\sqrt{a^2\sin^2 x - 1}} = -\frac{1}{a}\arcsin\frac{a\cos x}{\sqrt{a^2-1}}.$$

**2.** $$\int \frac{\cos x\,dx}{\sqrt{a^2\sin^2 x - 1}} = -\frac{1}{a}\ln\left(a\sin x + \sqrt{a^2\sin^2 x - 1}\,\right).$$

**3.** $$\int \frac{dx}{\sin x\sqrt{a^2\sin^2 x - 1}} = -\operatorname{arctg}\frac{\cos x}{\sqrt{a^2\sin^2 x - 1}}.$$

**4.** $$\int \frac{dx}{\cos x\sqrt{a^2\sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1}\,\sin x + \sqrt{a^2\sin^2 x - 1}}{\sqrt{a^2-1}\,\sin x - \sqrt{a^2\sin^2 x - 1}}.$$

**5.** $$\int \frac{\operatorname{tg} x\,dx}{\sqrt{a^2\sin^2 x - 1}} = \frac{1}{2\sqrt{a^2-1}}\ln\frac{\sqrt{a^2-1} + \sqrt{a^2\sin^2 x - 1}}{\sqrt{a^2-1} - \sqrt{a^2\sin^2 x - 1}}.$$

**6.** $$\int \frac{\operatorname{ctg} x\,dx}{\sqrt{a^2\sin^2 x - 1}} = -\arcsin\frac{1}{a\sin x}.$$

**5.44. Интегралы вида** $\displaystyle\int \operatorname{tg}^m x\,(a^2\operatorname{tg}^2 x \pm b^2)^{\pm n+1/2}\,dx.$

**1.** $$\int \operatorname{tg} x\,(a^2\operatorname{tg}^2 x + b^2)^{n+1/2}\,dx =$$
$$= \sqrt{a^2\operatorname{tg}^2 x + b^2}\sum_{k=0}^{n}\frac{(b^2-a^2)^k}{2n-2k+1}(a^2\operatorname{tg}^2 x + b^2)^{n-k} +$$
$$+\begin{cases} (-1)^{n+1}(a^2-b^2)^{n+1/2}\operatorname{arctg}\sqrt{\dfrac{a^2\operatorname{tg}^2 x + b^2}{a^2-b^2}} & [a^2 > b^2], \\[2ex] \dfrac{1}{2}(b^2-a^2)^{n+1/2}\ln\dfrac{\sqrt{b^2-a^2} - \sqrt{a^2\operatorname{tg}^2 x + b^2}}{\sqrt{b^2-a^2} + \sqrt{a^2\operatorname{tg}^2 x + b^2}} & [a^2 < b^2]. \end{cases}$$

**2.** $$\int \frac{\operatorname{tg} x\,dx}{(a^2\operatorname{tg}^2 x+b^2)^{n+1/2}} = \frac{1}{\sqrt{a^2\operatorname{tg}^2 x+b^2}}\sum_{k=0}^{n-1}\frac{(b^2-a^2)^{k-n}}{(2k+1)(a^2\operatorname{tg}^2 x+b^2)^k} + \\ + \begin{cases} \dfrac{(-1)^n}{(a^2-b^2)^{n+1/2}}\operatorname{arctg}\sqrt{\dfrac{a^2\operatorname{tg}^2 x+b^2}{a^2-b^2}} & [a^2>b^2], \\ \dfrac{1}{2(b^2-a^2)^{n+1/2}}\ln\dfrac{\sqrt{b^2-a^2}-\sqrt{a^2\operatorname{tg}^2 x+b^2}}{\sqrt{b^2-a^2}+\sqrt{a^2\operatorname{tg}^2 x+b^2}} & [a^2<b^2]. \end{cases}$$

**3.** $$\int \frac{dx}{\sqrt{a^2\operatorname{tg}^2 x+b^2}} = \frac{1}{2\sqrt{a^2-b^2}}\ln\frac{\sqrt{a^2-b^2}\,\operatorname{tg} x+\sqrt{a^2\operatorname{tg}^2 x+b^2}}{\sqrt{a^2-b^2}\,\operatorname{tg} x-\sqrt{a^2\operatorname{tg}^2 x+b^2}} \qquad [a^2>b^2],$$
$$= \frac{1}{\sqrt{b^2-a^2}}\operatorname{arctg}\frac{\sqrt{b^2-a^2}\,\operatorname{tg} x}{\sqrt{a^2\operatorname{tg}^2 x+b^2}} \qquad [a^2<b^2].$$

**4.** $$\int \frac{\operatorname{tg} x\,dx}{\sqrt{a^2\operatorname{tg}^2 x+b^2}} = \frac{1}{\sqrt{a^2-b^2}}\operatorname{arctg}\sqrt{\frac{a^2\operatorname{tg}^2 x+b^2}{a^2-b^2}} \qquad [a^2>b^2],$$
$$= \frac{1}{2\sqrt{b^2-a^2}}\ln\frac{\sqrt{b^2-a^2}-\sqrt{a^2\operatorname{tg}^2 x+b^2}}{\sqrt{b^2-a^2}+\sqrt{a^2\operatorname{tg}^2 x+b^2}} \qquad [a^2<b^2].$$

**5.** $$\int \frac{dx}{\sqrt{a^2\operatorname{tg}^2 x-b^2}} = \frac{1}{2\sqrt{a^2+b^2}}\ln\frac{\sqrt{a^2+b^2}\,\operatorname{tg} x+\sqrt{a^2\operatorname{tg}^2 x-b^2}}{\sqrt{a^2+b^2}\,\operatorname{tg} x-\sqrt{a^2\operatorname{tg}^2 x-b^2}}.$$

**6.** $$\int \frac{\operatorname{tg} x\,dx}{\sqrt{a^2\operatorname{tg}^2 x-b^2}} = \frac{1}{\sqrt{a^2+b^2}}\operatorname{arctg}\sqrt{\frac{a^2\operatorname{tg}^2 x-b^2}{a^2+b^2}}\,.$$

### 5.45. Интегралы вида $\int \operatorname{ctg}^m x(a^2\operatorname{ctg}^2 x\pm b^2)^{\pm n+1/2}\,dx$.

**1.** $$\int \operatorname{ctg} x(a^2\operatorname{ctg}^2 x+b^2)^{n+1/2}\,dx = \\ = -\sqrt{a^2\operatorname{ctg}^2 x+b^2}\sum_{k=0}^{n}\frac{(b^2-a^2)^k}{2n-2k+1}(a^2\operatorname{ctg}^2 x+b^2)^{n-k} - \\ - \begin{cases} (-1)^n(a^2-b^2)^{n+1/2}\operatorname{arcctg}\sqrt{\dfrac{a^2\operatorname{ctg}^2 x+b^2}{a^2-b^2}} & [a^2>b^2], \\ \dfrac{1}{2}(b^2-a^2)^{n+1/2}\ln\dfrac{\sqrt{b^2-a^2}-\sqrt{a^2\operatorname{tg}^2 x+b^2}}{\sqrt{b^2-a^2}+\sqrt{a^2\operatorname{ctg}^2 x+b^2}} & [a^2<b^2]. \end{cases}$$

**2.** $$\int \frac{\operatorname{ctg} x\,dx}{(a^2\operatorname{ctg}^2 x+b^2)^{n+1/2}} = \\ = -\frac{1}{\sqrt{a^2\operatorname{ctg}^2 x+b^2}}\sum_{k=0}^{n-1}\frac{(b^2-a^2)^{k-n}}{(2k+1)(a^2\operatorname{ctg}^2 x+b^2)^k} + \\ + \begin{cases} \dfrac{(-1)^n}{(a^2-b^2)^{n+1/2}}\operatorname{arcctg}\sqrt{\dfrac{a^2\operatorname{ctg}^2 x+b^2}{a^2-b^2}} & [a^2>b^2], \\ \dfrac{1}{2(b^2-a^2)^{n+1/2}}\ln\dfrac{\sqrt{b^2-a^2}+\sqrt{a^2\operatorname{ctg}^2 x+b^2}}{\sqrt{b^2-a^2}-\sqrt{a^2\operatorname{ctg}^2 x+b^2}} & [a^2<b^2]. \end{cases}$$

**3.** $$\int \frac{dx}{\sqrt{a^2\operatorname{ctg}^2 x + b^2}} = \frac{1}{2\sqrt{a^2-b^2}} \ln \frac{\sqrt{a^2-b^2}\operatorname{ctg} x - \sqrt{a^2\operatorname{ctg}^2 x + b^2}}{\sqrt{a^2-b^2}\operatorname{ctg} x + \sqrt{a^2\operatorname{ctg}^2 x + b^2}} \quad [a^2 > b^2],$$
$$= \frac{1}{\sqrt{b^2-a^2}} \operatorname{arcctg} \frac{\sqrt{b^2-a^2}\operatorname{ctg} x}{\sqrt{a^2\operatorname{ctg}^2 x + b^2}} \quad [a^2 < b^2].$$

**4.** $$\int \frac{\operatorname{ctg} x\,dx}{\sqrt{a^2\operatorname{ctg}^2 x + b^2}} = \frac{1}{\sqrt{a^2-b^2}} \operatorname{arcctg} \sqrt{\frac{a^2\operatorname{ctg}^2 x + b^2}{a^2-b^2}} \quad [a^2 > b^2],$$
$$= \frac{1}{2\sqrt{b^2-a^2}} \ln \frac{\sqrt{b^2-a^2} + \sqrt{a^2\operatorname{ctg}^2 x + b^2}}{\sqrt{b^2-a^2} - \sqrt{a^2\operatorname{ctg}^2 x + b^2}} \quad [a^2 < b^2].$$

**5.** $$\int \frac{dx}{\sqrt{a^2\operatorname{ctg}^2 x - b^2}} = \frac{1}{2\sqrt{a^2+b^2}} \ln \frac{\sqrt{a^2+b^2}\operatorname{ctg} x - \sqrt{a^2\operatorname{ctg}^2 x - b^2}}{\sqrt{a^2+b^2}\operatorname{ctg} x + \sqrt{a^2\operatorname{ctg}^2 x - b^2}}.$$

**6.** $$\int \frac{\operatorname{ctg} x\,dx}{\sqrt{a^2\operatorname{ctg}^2 x - b^2}} = \frac{1}{\sqrt{a^2+b^2}} \operatorname{arcctg} \sqrt{\frac{a^2\operatorname{ctg}^2 x - b^2}{a^2+b^2}}.$$

**5.46. Интегралы вида** $\int \operatorname{tg}^m x (a^2 - b^2\operatorname{tg}^2 x)^{\pm n+1/2}\,dx$.

**1.** $$\int \operatorname{tg} x (a^2 - b^2\operatorname{tg}^2 x)^{n+1/2}\,dx =$$
$$= \sqrt{a^2 - b^2\operatorname{tg}^2 x} \sum_{k=0}^{n} \frac{(a^2+b^2)^k}{2n-2k+1} (a^2 - b^2\operatorname{tg}^2 x)^{n-k} +$$
$$+ \frac{1}{2}(a^2+b^2)^{n+1/2} \ln \frac{\sqrt{a^2+b^2} - \sqrt{a^2 - b^2\operatorname{tg}^2 x}}{\sqrt{a^2+b^2} + \sqrt{a^2 - b^2\operatorname{tg}^2 x}}.$$

**2.** $$\int \frac{\operatorname{tg} x\,dx}{(a^2 - b^2\operatorname{tg}^2 x)^{n+1/2}} = \frac{1}{\sqrt{a^2 - b^2\operatorname{tg}^2 x}} \sum_{k=0}^{n-1} \frac{(a^2+b^2)^{k-n}}{(2k+1)(a^2 - b^2\operatorname{tg}^2 x)^k} +$$
$$+ \frac{1}{2(a^2+b^2)^{n+1/2}} \ln \frac{\sqrt{a^2+b^2} - \sqrt{a^2 - b^2\operatorname{tg}^2 x}}{\sqrt{a^2+b^2} + \sqrt{a^2 - b^2\operatorname{tg}^2 x}}.$$

**3.** $$\int \frac{dx}{\sqrt{a^2 - b^2\operatorname{tg}^2 x}} = \frac{1}{\sqrt{a^2+b^2}} \operatorname{arctg} \frac{\sqrt{a^2+b^2}\operatorname{tg} x}{\sqrt{a^2 - b^2\operatorname{tg}^2 x}}.$$

**4.** $$\int \frac{\operatorname{tg} x\,dx}{\sqrt{a^2 - b^2\operatorname{tg}^2 x}} = \frac{1}{2\sqrt{a^2+b^2}} \ln \frac{\sqrt{a^2+b^2} - \sqrt{a^2 - b^2\operatorname{tg}^2 x}}{\sqrt{a^2+b^2} + \sqrt{a^2 - b^2\operatorname{tg}^2 x}}.$$

**5.47. Интегралы вида** $\int \operatorname{ctg}^m x (a^2 - b^2\operatorname{ctg}^2 x)^{\pm n+1/2}\,dx$.

**1.** $$\int \operatorname{ctg} x (a^2 - b^2\operatorname{ctg}^2 x)^{n+1/2}\,dx =$$
$$= -\sqrt{a^2 - b^2\operatorname{ctg}^2 x} \sum_{k=0}^{n} \frac{(a^2+b^2)^k}{2n-2k+1} (a^2 - b^2\operatorname{ctg}^2 x)^{n-k} -$$
$$- \frac{1}{2}(a^2+b^2)^{n+1/2} \ln \frac{\sqrt{a^2+b^2} - \sqrt{a^2 - b^2\operatorname{ctg}^2 x}}{\sqrt{a^2+b^2} + \sqrt{a^2 - b^2\operatorname{ctg}^2 x}}.$$

2. $$\int \frac{\operatorname{ctg} x\,dx}{(a^2 - b^2 \operatorname{ctg}^2 x)^{n+1/2}} = -\frac{1}{\sqrt{a^2 - b^2 \operatorname{ctg}^2 x}} \times \sum_{k=0}^{n-1} \frac{(a^2+b^2)^{k-n}}{(2k+1)(a^2 - b^2 \operatorname{ctg}^2 x)^k} - \frac{1}{2(a^2+b^2)^{n+1/2}} \ln \frac{\sqrt{a^2+b^2} - \sqrt{a^2 - b^2 \operatorname{ctg}^2 x}}{\sqrt{a^2+b^2} + \sqrt{a^2 - b^2 \operatorname{ctg}^2 x}}.$$

3. $$\int \frac{dx}{\sqrt{a^2 - b^2 \operatorname{ctg}^2 x}} = \frac{1}{\sqrt{a^2+b^2}} \operatorname{arcctg} \frac{\sqrt{a^2+b^2}\,\operatorname{ctg} x}{\sqrt{a^2 - b^2 \operatorname{ctg}^2 x}}.$$

4. $$\int \frac{\operatorname{ctg} x\,dx}{\sqrt{a^2 - b^2 \operatorname{ctg}^2 x}} = \frac{1}{2\sqrt{a^2+b^2}} \ln \frac{\sqrt{a^2+b^2} + \sqrt{a^2 - b^2 \operatorname{ctg}^2 x}}{\sqrt{a^2+b^2} - \sqrt{a^2 - b^2 \operatorname{ctg}^2 x}}.$$

### **5.48.** Интегралы вида $\int x^p \sin^q x\,dx$.

1. $$\int x^p \sin^q x\,dx = \frac{x^{p-1}}{q^2} \sin^{q-1} x \times (p \sin x - qx \cos x) + \frac{q-1}{q} \int x^p \sin^{q-2} x\,dx - \frac{p(p-1)}{q^2} \int x^{p-2} \sin^q x\,dx.$$

2. $$\int x^p \sin x\,dx = -x^p \cos x + p \int x^{p-1} \cos x\,dx.$$

3. $$\int x^m \sin^{2n} x\,dx = C_{2n}^n \frac{x^{m+1}}{2^{2n}(m+1)} + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^{n+k} C_{2n}^k \int x^m \cos(2n-2k)x\,dx.$$

4. $$\int x^m \sin^{2n+1} x\,dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n+k} C_{2n+1}^k \int x^m \sin(2n-2k+1)x\,dx.$$

5. $$\int x^m \sin x\,dx = -\sum_{k=0}^{m} k!\,C_m^k x^{m-k} \cos\left(x + \frac{k\pi}{2}\right).$$

6. $$\int x^{2m} \sin x\,dx = (2m)!\left[\sum_{k=0}^{m-1} (-1)^k \frac{x^{2m-2k-1}}{(2m-2k-1)!} \sin x - \sum_{k=0}^{m} (-1)^k \frac{x^{2m-2k}}{(2m-2k)!} \cos x\right].$$

7. $$\int x^{2m+1} \sin x\,dx = (2m+1)!\left[\sum_{k=0}^{m} (-1)^k \frac{x^{2m-2k}}{(2m-2k)!} \sin x - \sum_{k=0}^{m} (-1)^k \frac{x^{2m-2k+1}}{(2m-2k+1)!} \cos x\right].$$

8. $$\int x \sin^{2n} x\,dx = \frac{(2n-1)!!}{(2n)!!} \frac{x^2}{2} + \frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2^{k+1} n(n-1)\ldots(n-k)} \times \left(\frac{\sin^{2n-2k} x}{2n-2k} - x \sin^{2n-2k-1} x \cos x\right).$$

**9.** $$\int x\sin^{2n+1}x\,dx = \frac{1}{n+1}\sum_{k=0}^{n}\frac{2^k(n+1)n\ldots(n-k+1)}{(2n+1)(2n-1)\ldots(2n-2k+1)}\times$$
$$\times\left(\frac{\sin^{2n-2k+1}x}{2n-2k+1} - x\sin^{2n-2k}x\cos x\right).$$

**10.** $$\int x\sin x\,dx = \sin x - x\cos x.$$

**11.** $$\int x^2\sin x\,dx = 2x\sin x - (x^2-2)\cos x.$$

**12.** $$\int x^3\sin x\,dx = (3x^2-6)\sin x - (x^3-6x)\cos x.$$

**13.** $$\int x^m\sin^2 x\,dx = \frac{x^{m+1}}{2(m+1)} +$$
$$+\frac{m!}{4}\left[\sum_{k=0}^{[m/2]}\frac{(-1)^{k+1}x^{m-2k}}{2^{2k}(m-2k)!}\sin 2x + \sum_{k=0}^{[(m-1)/2]}\frac{(-1)^{k+1}x^{m-2k-1}}{2^{2k+1}(m-2k-1)!}\cos 2x\right].$$

**14.** $$\int x\sin^2 x\,dx = \frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{1}{8}\cos 2x.$$

**15.** $$\int x^2\sin^2 x\,dx = \frac{x^3}{6} - \left(\frac{x^2}{4} - \frac{1}{8}\right)\sin 2x - \frac{x}{4}\cos 2x.$$

**16.** $$\int x^3\sin^2 x\,dx = \frac{x^4}{8} - \left(\frac{x^3}{4} - \frac{3x}{8}\right)\sin 2x - \left(\frac{3x^2}{8} - \frac{3}{16}\right)\cos 2x.$$

**17.** $$\int x^m\sin^3 x\,dx = \frac{m!}{4}\left[\sum_{k=0}^{[m/2]}\frac{(-1)^k x^{m-2k}}{(m-2k)!}\left(\frac{1}{3^{2k+1}}\cos 3x - 3\cos x\right) -\right.$$
$$\left. - \sum_{k=0}^{[(m-1)/2]}\frac{(-1)^k x^{m-2k-1}}{(m-2k-1)!}\left(\frac{1}{3^{2k+2}}\sin 3x - 3\sin x\right)\right].$$

**18.** $$\int x\sin^3 x\,dx = \frac{3}{4}\sin x - \frac{1}{36}\sin 3x - \frac{3}{4}x\cos x + \frac{x}{12}\cos 3x.$$

## 5.49. Интегралы вида $\int \frac{\sin^q x}{x^p}\,dx$.

Неопределенные интегралы, приведенные в этом разделе, при $p > 0$ не выражаются через элементарные функции, но могут быть при $p = 1, 2, 3, \ldots$ выражены через специальные функции — интегральный синус $\operatorname{si}(x)$ и интегральный косинус $\operatorname{ci}(x)$, которые определяются формулами

$$\operatorname{si}(x) = -\int_x^{\infty}\frac{\sin t}{t}\,dt,$$

$$\operatorname{ci}(x) = -\int_x^{\infty}\frac{\cos t}{t}\,dt, \qquad x > 0,$$

и при $p = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ — через синус-интеграл Френеля $\mathrm{S}(x)$ и косинус-интеграл Френеля $\mathrm{C}(x)$, определяемые формулами

$$\mathrm{S}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}}\, dt, \qquad \mathrm{C}(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}}\, dt.$$

**1.** $$\int \frac{\sin^q x}{x^p}\, dx = -\frac{\sin^q x}{(p-1)x^{p-1}} - \frac{q \sin^{q-1} x \cos x}{(p-1)(p-2)x^{p-2}} + \frac{q(q-1)}{(p-1)(p-2)} \int \frac{\sin^{q-2} x}{x^{p-2}}\, dx - \frac{q^2}{(p-1)(p-2)} \int \frac{\sin^q x}{x^{p-2}}\, dx.$$

**2.** $$\int \frac{\sin x}{x^p}\, dx = -\frac{\sin x}{(p-1)x^{p-1}} + \frac{1}{p-1} \int \frac{\cos x}{x^{p-1}}\, dx.$$

**3.** $$= -\frac{\sin x}{(p-1)x^{p-1}} - \frac{\cos x}{(p-1)(p-2)x^{p-2}} - \frac{1}{(p-1)(p-2)} \int \frac{\sin x}{x^{p-2}}\, dx.$$

**4.** $$\int \frac{\sin x}{x^{2m}}\, dx = \frac{(-1)^{m+1}}{(2m-1)!\, x} \left[ \sum_{k=0}^{m-2} (-1)^k \frac{(2k+1)!}{x^{2k+1}} \cos x - \sum_{k=0}^{m-1} (-1)^k \frac{(2k)!}{x^{2k}} \sin x \right] + \frac{(-1)^{m+1}}{(2m-1)!} \operatorname{ci}(x).$$

**5.** $$\int \frac{\sin x}{x^{2m+1}}\, dx = \frac{(-1)^m}{(2m)!\, x} \left[ \sum_{k=0}^{m-1} (-1)^k \frac{(2k)!}{x^{2k}} \cos x + \sum_{k=0}^{m-1} (-1)^k \frac{(2k+1)!}{x^{2k+1}} \sin x \right] + \frac{(-1)^m}{(2m)!} \operatorname{si}(x).$$

**6.** $$\int \frac{\sin^{2n} x}{x}\, dx = \frac{C_{2n}^n}{2^{2n}} \ln x + \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^{n+k} C_{2n}^k \operatorname{ci}((2n-2k)x).$$

**7.** $$\int \frac{\sin^{2n+1} x}{x}\, dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n+k} C_{2n+1}^k \operatorname{si}((2n-2k+1)x).$$

**8.** $$\int \frac{\sin^{2n} x}{x^2}\, dx = -\frac{C_{2n}^n}{2^{2n} x} - \frac{(-1)^n}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k C_{2n}^k \left[ \frac{\cos(2n-2k)x}{x} + (2n-2k) \operatorname{si}((2n-2k)x) \right].$$

**9.** $$\int \frac{\sin^{2n+1} x}{x^2}\, dx = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^{n+k+1} C_{2n+1}^k \times \left[ \frac{\sin(2n-2k+1)x}{x} - (2n-2k+1) \operatorname{ci}((2n-2k+1)x) \right].$$

**10.** $$\int \frac{\sin x}{x}\, dx = \operatorname{si} x.$$

**11.** $$\int \frac{\sin x}{x^2}\, dx = -\frac{\sin x}{x} + \operatorname{ci}(x).$$

**12.** $\displaystyle\int \frac{\sin x}{x^3}\,dx = -\frac{\sin x}{2x^2} - \frac{\cos x}{2x} - \frac{1}{2}\,\mathrm{si}\,(x).$

**13.** $\displaystyle\int \frac{\sin^2 x}{x}\,dx = \frac{1}{2}\ln x - \frac{1}{2}\,\mathrm{ci}\,(2x).$

**14.** $\displaystyle\int \frac{\sin^3 x}{x}\,dx = \frac{3}{4}\,\mathrm{si}\,(x) - \frac{1}{4}\,\mathrm{si}\,(3x).$

**5.50. Интегралы вида** $\displaystyle\int \frac{x^p}{\sin^q x}\,dx.$

**1.** $$\int \frac{x^p}{\sin^q x}\,dx = -\frac{x^{p-1}[p\sin x + (q-2)x\cos x]}{(q-1)(q-2)\sin^{q-1} x} + \frac{q-2}{q-1}\int \frac{x^p\,dx}{\sin^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int \frac{x^{p-2}\,dx}{\sin^{q-2} x}.$$

**2.** $$\int \frac{x^p}{\sin x}\,dx = \frac{x^p}{p} - 2\sum_{k=1}^{\infty}(-1)^k\frac{(2^{2k-1}-1)}{(2k)!\,(2k+p)}\,B_{2k}x^{2k+p} \qquad [p>0;\ |x|<\pi].$$

**3.** $$\int \frac{x^p}{\sin^2 x}\,dx = -x^p\,\mathrm{ctg}\,x + \frac{p}{p-1}x^{p-1} + p\sum_{k=1}^{\infty}(-1)^k\frac{2^{2k}B_{2k}}{(2k)!\,(2k+p-1)}\,x^{2k+p-1} \qquad [p>1;\ |x|<\pi].$$

**4.** $$\int \frac{x}{\sin^q x}\,dx = -\frac{x\cos x}{(q-1)\sin^{q-1} x} - \frac{1}{(q-1)(q-2)\sin^{q-2} x} + \frac{q-2}{q-1}\int \frac{x\,dx}{\sin^{q-2} x}.$$

**5.** $$\int \frac{x\,dx}{\sin^{2n} x} = -\frac{1}{2n}\sum_{k=1}^{n-1}\frac{2n(2n-2)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+3)(2n-2k+1)}\times \frac{\sin x + (2n-2k)x\cos x}{(2n-2k)\sin^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!}\,(\ln|\sin x| - x\,\mathrm{ctg}\,x).$$

**6.** $$\int \frac{x\,dx}{\sin^{2n+1} x} = -\frac{1}{2n+1}\sum_{k=0}^{n-1}\frac{(2n+1)\ldots(2n-2k+1)}{2n(2n-2)\ldots(2n-2k+2)(2n-2k)}\times \frac{\sin x + (2n-2k-1)x\cos x}{(2n-2k-1)\sin^{2n-2k} x} + \frac{(2n-1)!!}{2^n n!}\int \frac{x\,dx}{\sin x}.$$

**7.** $$\int \frac{dx}{x^p\sin^q x} = \frac{p}{(q-1)(q-2)x^{p+1}\sin^{q-2} x} - \frac{\cos x}{(q-1)x^p\sin^{q-1} x} + \frac{q-2}{q-1}\int \frac{dx}{x^p\sin^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)}\int \frac{dx}{x^{p+2}\sin^{q-2} x}.$$

**8.** $$\int \frac{dx}{x^n\sin x} = -\frac{1}{nx^n} + [1+(-1)^n]\frac{2^{n-1}-1}{n!}\,|B_n|\ln|x| + 2\sum_{\substack{k=1\\ k\neq n/2}}^{\infty}\frac{2^{2k-1}-1}{(2k)!(2k-n)}\,|B_{2k}|x^{2k-n} \qquad [n\geqslant 1;\ |x|<\pi].$$

**9.** $$\int \frac{dx}{x^n \sin^2 x} = -\frac{\operatorname{ctg} x}{x^n} + \frac{n}{(n+1)x^{n+1}} + [1-(-1)^n]\frac{2^n n}{(n+1)!}\,|B_{n+1}|\ln|x| + \frac{n}{x^{n+1}} \sum_{\substack{k=1\\ k\neq(n+1)/2}}^{\infty} \frac{|B_{2k}|}{(2k)!(2k-n-1)}\,(2x)^{2k} \qquad [|x|<\pi].$$

**10.** $$\int \frac{x\,dx}{\sin x} = x + 2\sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{(2k+1)!}\,|B_{2k}|x^{2k+1}.$$

**11.** $$\int \frac{x^2\,dx}{\sin x} = \frac{x^2}{2} + \sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{(2k)!(k+1)}\,|B_{2k}|x^{2k+2}.$$

**12.** $$\int \frac{x\,dx}{\sin^2 x} = -x\operatorname{ctg} x + \ln|\sin x|.$$

**13.** $$\int \frac{x\,dx}{\sin^3 x} = -\frac{\sin x + x\cos x}{2\sin^2 x} + \frac{1}{2}\int \frac{x\,dx}{\sin x}.$$

**14.** $$\int \frac{x\,dx}{\sin^4 x} = -\frac{x\cos x}{3\sin^3 x} - \frac{1}{6\sin^2 x} - \frac{2}{3}\,x\operatorname{ctg} x + \frac{2}{3}\ln|\sin x|.$$

### 5.51. Интегралы вида $\int x^p \cos^q x\,dx$.

**1.** $$\int x^p \cos^q x\,dx = \frac{x^{p-1}}{q^2}\cos^{q-1} x \times (p\cos x + qx\sin x) + \frac{q-1}{q}\int x^p\cos^{q-2} x\,dx - \frac{p(p-1)}{q^2}\int x^{p-2}\cos^q x\,dx.$$

**2.** $$\int x^p \cos x\,dx = x^p \sin x - p\int x^{p-1}\sin x\,dx.$$

**3.** $$\int x^m \cos^{2n} x\,dx = C_{2n}^n \frac{x^{m+1}}{2^{2n}(m+1)} + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \int x^m \cos(2n-2k)x\,dx.$$

**4.** $$\int x^m \cos^{2n+1} x\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \int x^m \cos(2n-2k+1)x\,dx.$$

**5.** $$\int x^m \cos x\,dx = \sum_{k=0}^{n} k!C_m^k x^{m-k} \sin\left(x + \frac{k\pi}{2}\right).$$

**6.** $$\int x^{2m}\cos x\,dx = (2m)!\left[\sum_{k=0}^{m}(-1)^k \frac{x^{2m-2k}}{(2m-2k)!}\sin x + \sum_{k=0}^{m-1}(-1)^k \frac{x^{2m-2k-1}}{(2m-2k-1)!}\cos x\right].$$

**7.** $$\int x^{2m+1}\cos x\,dx = (2m+1)!\left[\sum_{k=0}^{m}(-1)^k \frac{x^{2m-2k+1}}{(2m-2k+1)!}\sin x + \sum_{k=0}^{m}(-1)^k \frac{x^{2m-2k}}{(2m-2k)!}\cos x\right].$$

**8.** $$\int x\cos^{2n} x\,dx = \frac{(2n-1)!!}{(2n)!!}\,\frac{x^2}{2} + \frac{1}{2n+1} \times \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2^{k+1}n(n-1)\ldots(n-k)}\left(\frac{\cos x^{2n-2k} x}{2n-2k} + x\sin x\cos^{2n-2k-1} x\right).$$

**9.** $$\int x\cos^{2n+1}x\,dx = \frac{1}{n+1}\sum_{k=0}^{n}\frac{2^k(n+1)n\ldots(n-k+1)}{(2n+1)(2n-1)\ldots(2n-2k+1)}\times$$
$$\times\left(\frac{\cos^{2n-2k+1}x}{2n-2k+1}+x\sin x\cos^{2n-2k}x\right).$$

**10.** $$s\int x\cos x\,dx = \cos x + x\sin x.$$

**11.** $$\int x^2\cos x\,dx = 2x\cos x + (x^2-2)\sin x.$$

**12.** $$\int x^3\cos x\,dx = (3x^2-6)\cos x + (x^3-6x)\sin x.$$

**13.** $$\int x^m\cos^2 x\,dx = \frac{x^{m+1}}{2(m+1)} -$$
$$-\frac{m!}{4}\left[\sum_{k=0}^{[m/2]}\frac{(-1)^{k+1}x^{m-2k}}{2^{2k}(m-2k)!}\sin 2x + \sum_{k=0}^{[(m-1)/2]}\frac{(-1)^{k+1}x^{m-2k-1}}{2^{2k+1}(m-2k-1)!}\cos 2x\right].$$

**14.** $$\int x\cos^2 x\,dx = \frac{x^2}{4}+\frac{x}{4}\sin 2x+\frac{1}{8}\cos 2x.$$

**15.** $$\int x^2\cos^2 x\,dx = \frac{x^3}{6}+\left(\frac{x^2}{4}-\frac{1}{8}\right)\sin 2x+\frac{x}{4}\cos 2x.$$

**16.** $$\int x^3\cos^2 x\,dx = \frac{x^4}{8}+\left(\frac{x^3}{4}-\frac{3x}{8}\right)\sin 2x+\left(\frac{3x^2}{8}-\frac{3}{16}\right)\cos 2x.$$

**17.** $$\int x^m\cos^3 x\,dx = \frac{m!}{4}\left[\sum_{k=0}^{[m/2]}\frac{(-1)^k x^{m-2k}}{(m-2k)!}\left(\frac{1}{3^{2k+1}}\sin 3x+3\sin x\right)+\right.$$
$$\left.+\sum_{k=0}^{[(m-1)/2]}\frac{(-1)^k x^{m-2k-1}}{(m-2k-1)!}\left(\frac{1}{3^{2k+2}}\cos 3x+3\cos x\right)\right].$$

**18.** $$\int x\cos^3 x\,dx = \frac{3}{4}\cos x+\frac{1}{36}\cos 3x+\frac{3}{4}x\sin x+\frac{x}{12}\sin 3x.$$

### 5.52. Интегралы вида $\int \frac{\cos^q x}{x^p}\,dx$ (см. замечание в начале 5.49).

**1.** $$\int\frac{\cos^q x}{x^p}\,dx = -\frac{\cos^q x}{(p-1)x^{p-1}}+\frac{q\sin x\cos^{q-1}x}{(p-1)(p-2)x^{p-2}}+$$
$$+\frac{q(q-1)}{(p-1)(p-2)}\int\frac{\cos^{q-2}x}{x^{p-2}}\,dx-\frac{q^2}{(p-1)(p-2)}\int\frac{\cos^q x}{x^{p-2}}\,dx.$$

**2.** $$\int\frac{\cos x}{x^p}\,dx = -\frac{\cos x}{(p-1)x^{p-1}}-\frac{1}{p-1}\int\frac{\sin x}{x^{p-1}}\,dx.$$

**3.** $$= -\frac{\cos x}{(p-1)x^{p-1}}+\frac{\sin x}{(p-1)(p-2)x^{p-2}}-\frac{1}{(p-1)(p-2)}\int\frac{\cos x}{x^{p-2}}\,dx.$$

**4.** $$\int\frac{\cos x}{x^{2m}}\,dx = \frac{(-1)^m}{(2m-1)!\,x}\left[\sum_{k=0}^{m-2}(-1)^k\frac{(2k+1)!}{x^{2k+1}}\sin x+\right.$$
$$\left.+\sum_{k=0}^{m-1}(-1)^k\frac{(2k)!}{x^{2k}}\cos x\right]+\frac{(-1)^m}{(2m-1)!}\,\mathrm{si}\,(x).$$

5. $$\int \frac{\cos x}{x^{2m+1}}\,dx = \frac{(-1)^m}{(2m)!\,x}\left[\sum_{k=0}^{m-1}(-1)^k\frac{(2k+1)!}{x^{2k+1}}\cos x - \right.$$
$$\left. - \sum_{k=0}^{m-1}(-1)^k\frac{(2k)!}{x^{2k}}\sin x\right] + \frac{(-1)^m}{(2m)!}\,\mathrm{ci}\,(x).$$

6. $$\int \frac{\cos^{2n} x}{x}\,dx = \frac{C_{2n}^n}{2^{2n}}\ln x + \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k\,\mathrm{ci}\,((2n-2k)x).$$

7. $$\int \frac{\cos^{2n+1} x}{x}\,dx = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k\,\mathrm{ci}\,((2n-2k+1)x).$$

8. $$\int \frac{\cos^{2n} x}{x^2}\,dx = -\frac{C_{2n}^n}{2^{2n}x} -$$
$$-\frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k\left[\frac{\cos(2n-2k)x}{x} + (2n-2k)\,\mathrm{si}\,((2n-2k)x)\right].$$

9. $$\int \frac{\cos^{2n+1} x}{x^2}\,dx = -\frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \times$$
$$\times\left[\frac{\cos(2n-2k+1)x}{x} + (2n-2k+1)\,\mathrm{si}\,((2n-2k+1)x)\right].$$

10. $$\int \frac{\cos x}{x}\,dx = \mathrm{ci}\,x.$$

11. $$\int \frac{\cos x}{x^2}\,dx = -\frac{\cos x}{x} - \mathrm{si}\,(x).$$

12. $$\int \frac{\cos x}{x^3}\,dx = -\frac{\cos x}{2x^2} + \frac{\sin x}{2x} - \frac{1}{2}\,\mathrm{ci}\,(x).$$

13. $$\int \frac{\cos^2 x}{x}\,dx = \frac{1}{2}\ln x + \frac{1}{2}\,\mathrm{ci}\,(2x).$$

14. $$\int \frac{\cos^3 x}{x}\,dx = \frac{3}{4}\,\mathrm{ci}\,(x) + \frac{1}{4}\,\mathrm{ci}\,(3x).$$

### 5.53. Интегралы вида $\int \frac{x^p}{\cos^q x}\,dx$.

1. $$\int \frac{x^p\,dx}{\cos^q x} = -\frac{x^{p-1}[p\cos x - (q-2)x\sin x]}{(q-1)(q-2)\cos^{q-1} x} +$$
$$+\frac{q-2}{q-1}\int\frac{x^p\,dx}{\cos^{q-2} x} + \frac{p(p-1)}{(q-1)(q-2)}\int\frac{x^{p-2}\,dx}{\cos^{q-2} x}.$$

2. $$\int \frac{x^p\,dx}{\cos x} = \sum_{k=0}^{\infty}\frac{|E_{2k}|x^{2k+p+1}}{(2k)!\,(2k+p+1)} \qquad [p>0;\ |x|<\pi/2].$$

3. $$\int \frac{x^p\,dx}{\cos^2 x} = x^p\,\mathrm{tg}\,x - p\sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)|B_{2k}|}{(2k)!\,(2k+p-1)}\,x^{2k+p-1} \qquad [p>1;\ |x|<\pi/2].$$

**4.** $$\int \frac{x\,dx}{\cos^{2n} x} = \frac{1}{2n} \sum_{k=1}^{n-1} \frac{2n(2n-2)\ldots(2n-2k+2)}{(2n-1)(2n-3)\ldots(2n-2k+3)(2n-2k+1)} \times \\ \times \frac{(2n-2k)x\sin x - \cos x}{(2n-2k)\cos^{2n-2k+1} x} + \frac{2^{n-1}(n-1)!}{(2n-1)!!}\,(x\,\mathrm{tg}\,x + \ln|\cos x|).$$

**5.** $$\int \frac{x\,dx}{\cos^{2n+1} x} = \frac{1}{2n+1} \sum_{k=0}^{n-1} \frac{(2n+1)(2n-1)\ldots(2n-2k+1)}{2n(2n-2)\ldots(2n-2k+2)(2n-2k)} \times \\ \times \frac{(2n-2k+1)x\sin x - \cos x}{(2n-2k-1)\cos^{2n-2k} x} + \frac{(2n-1)!!}{2^n n} \int \frac{x\,dx}{\cos x}.$$

**6.** $$\int \frac{dx}{x^p \cos^q x} = \frac{p}{(q-1)(q-2)x^{p+1}\cos^{q-2} x} + \frac{\sin x}{(q-1)x^p\cos^{q-1} x} + \\ + \frac{q-2}{q-1}\int \frac{dx}{x^p\cos^{q-2} x} + \frac{p(p+1)}{(q-1)(q-2)}\int \frac{dx}{x^{p+2}\cos^{q-2} x}.$$

**7.** $$\int \frac{dx}{x^n \cos x} = \frac{1}{2}[1-(-1)^n]\frac{|E_{n-1}|}{(n-1)!}\,\ln|x| + \\ + \sum_{\substack{k=0\\ k\neq(n-1)/2}}^{\infty} \frac{|E_{2k}|x^{2k-n+1}}{(2k)!(2k-n+1)} \qquad [|x| < \pi/2].$$

**8.** $$\int \frac{dx}{x^n \cos^2 x} = \frac{\mathrm{tg}\,x}{x^n} + [1-(-1)^n]\frac{2^n n}{(n+1)!}\,(2^{n+1}-1)|B_{n+1}|\ln|x| + \\ + \frac{n}{x^{n+1}} \sum_{\substack{k=1\\ k\neq(n+1)/2}}^{\infty} \frac{(2^{2k}-1)|B_{2k}|}{(2k)!(2k-n-1)}\,(2x)^{2k} \qquad [|x| < \pi/2].$$

**9.** $$\int \frac{x\,dx}{\cos x} = \sum_{k=0}^{\infty} \frac{|E_{2k}|x^{2k+2}}{(2k)!\,(2k+2)}.$$

**10.** $$\int \frac{x\,dx}{\cos^2 x} = x\,\mathrm{tg}\,x + \ln|\cos x|.$$

**11.** $$\int \frac{x\,dx}{\cos^3 x} = \frac{x\sin x - \cos x}{2\cos^2 x} + \frac{1}{2}\int \frac{x\,dx}{\cos x}.$$

**12.** $$\int \frac{x\,dx}{\cos^4 x} = \frac{x\sin x}{3\cos^3 x} - \frac{1}{6\cos^2 x} + \frac{2}{3}\,x\,\mathrm{tg}\,x - \frac{2}{3}\ln|\cos x|.$$

**5.54.** Интегралы вида $\int x^p\,\mathrm{tg}^q\,x\,dx$.

**1.** $$\int x^p\,\mathrm{tg}\,x\,dx = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k-1}-1)}{(2k)!\,(2k+p)}\,|B_{2k}|x^{2k+p} \qquad [p \geqslant -1;\ |x| < \pi/2].$$

**2.** $$\int x^p\,\mathrm{tg}^{2n}\,x\,dx = (-1)^n\frac{x^{p+1}}{p+1} + \sum_{k=0}^{n-1}(-1)^k C_n^k \int \frac{x^p\,dx}{\cos^{2n-2k} x} \qquad [\text{см. } 5.53].$$

**3.** $$\int x^p \operatorname{tg}^{2n+1} x\,dx = \sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} C_n^k \frac{x^p}{\cos^{2n-2k} x} + (-1)^n \int x^p \operatorname{tg} x\,dx - p\sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} C_n^k \int \frac{x^{p-1}\,dx}{\cos^{2n-2k} x}$$ [см. 5.53].

**4.** $$\int x \operatorname{tg}^2 x\,dx = x \operatorname{tg} x + \ln|\cos x| - \frac{x^2}{2}.$$

**5.55.** Интегралы вида $\int x^p \operatorname{ctg}^q x\,dx$.

**1.** $$\int x^p \operatorname{ctg} x\,dx = -\sum_{k=0}^{\infty} \frac{2^{2k}|B_{2k}|}{(2k)!\,(2k+p)} x^{2k+p}$$ [$p \geqslant 1;\ |x| < \pi$].

**2.** $$\int x^p \operatorname{ctg}^{2n} x\,dx = (-1)^n \frac{x^{p+1}}{p+1} + \sum_{k=0}^{n-1} (-1)^k C_n^k \int \frac{x^p\,dx}{\sin^{2n-2k} x}$$ [см. 5.50].

**3.** $$\int x^p \operatorname{ctg}^{2n+1} x\,dx = -\sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} C_n^k \frac{x^p}{\sin^{2n-2k} x} + (-1)^n \int x^p \operatorname{ctg} x\,dx + p\sum_{k=0}^{n-1} \frac{(-1)^k}{2n-2k} C_n^k \int \frac{x^{p-1}\,dx}{\sin^{2n-2k} x}$$ [см. 5.50].

**4.** $$\int x \operatorname{ctg}^2 x\,dx = -x \operatorname{ctg} x + \ln|\sin x| - \frac{x^2}{2}.$$

**5.56.** Интегралы вида $\int x^r \sin^p x \cos^q x\,dx$.

**1.** $$\int x^r \sin^p x \cos^q x\,dx = \frac{1}{(p+q)^2}\Big[(p+q)x^r \sin^{p+1} x \cos^{q-1} x + r x^{r-1} \sin^p x \cos^q x - r(r-1)\int x^{r-2} \sin^p x \cos^q x\,dx - rp\int x^{r-1} \sin^{p-1} x \cos^{q-1} x\,dx + (q-1)(p+q)\int x^r \sin^p x \cos^{q-2} x\,dx\Big].$$

**2.** $$= \frac{1}{(p+q)^2}\Big[-(p+q)x^r \sin^{p-1} x \cos^{q+1} x + r x^{r-1} \sin^p x \cos^q x - r(r-1)\int x^{r-2} \sin^p x \cos^q x\,dx + rq\int x^{r-1} \sin^{p-1} x \cos^{q-1} x\,dx + (p-1)(p+q)\int x^r \sin^{p-2} x \cos^q x\,dx\Big].$$

**3.** $$\int x \sin^p x \cos^q x\,dx = \frac{1}{p+q}\Big[x \sin^{p+1} x \cos^{q-1} x - \int \sin^{p+1} x \cos^{q-1} x\,dx + (q-1)\int x \sin^p x \cos^{q-2} x\,dx\Big].$$

**4.** $$= \frac{1}{p+q}\Big[-x \sin^{p-1} x \cos^{q+1} x + \int \sin^{p-1} x \cos^{q+1} x\,dx + (p-1)\int x \sin^{p-2} x \cos^q x\,dx\Big].$$

**5.** $\displaystyle\int x^p \frac{\sin^{2m} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k C_m^k \int \frac{x^p\,dx}{\cos^{n-2k} x}$ [см. 5.53].

**6.** $\displaystyle\int x^p \frac{\sin^{2m+1} x}{\cos^n x}\,dx = \sum_{k=0}^{m} (-1)^k C_m^k \int \frac{x^p \sin x}{\cos^{n-2k} x}\,dx.$

**7.** $\displaystyle\int x^p \frac{\sin x}{\cos^n x}\,dx = \frac{x^p}{(n-1)\cos^{n-1} x} - \frac{p}{n-1}\int \frac{x^{p-1}}{\cos^{n-1} x}\,dx$ [см. 5.53].

**8.** $\displaystyle\int x^p \frac{\cos^{2m} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k C_m^k \int \frac{x^p}{\sin^{n-2k} x}\,dx$ [см. 5.50].

**9.** $\displaystyle\int x^p \frac{\cos^{2m+1} x}{\sin^n x}\,dx = \sum_{k=0}^{m} (-1)^k C_m^k \int \frac{x^p \cos x}{\sin^{n-2k} x}\,dx.$

**10.** $\displaystyle\int x^p \frac{\cos x}{\sin^n x}\,dx = -\frac{x^p}{(n-1)\sin^{n-1} x} + \frac{p}{n-1}\int \frac{x^{p-1}}{\sin^{n-1} x}\,dx$ [см. 5.50].

**11.** $\displaystyle\int \frac{x\cos x}{\sin^2 x}\,dx = -\frac{x}{\sin x} + \ln\left|\operatorname{tg}\frac{x}{2}\right|.$

**12.** $\displaystyle\int \frac{x\sin x}{\cos^2 x}\,dx = \frac{x}{\cos x} - \ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|.$

**5.57. Интегралы вида** $\displaystyle\int \frac{x^p \sin^m x \cos^n x}{(a + b\cos x + c\sin x)^q}\,dx.$

**1.** $\displaystyle\int \frac{x^p \cos x\,dx}{(a+b\sin x)^q} = -\frac{x^p}{(q-1)b(a+b\sin x)^{q-1}} + \frac{p}{(q-1)b}\int \frac{x^{p-1}\,dx}{(a+b\sin x)^{q-1}}.$

**2.** $\displaystyle\int \frac{x^p \sin x\,dx}{(a+b\cos x)^q} = \frac{x^p}{(q-1)b(a+b\cos x)^{q-1}} - \frac{p}{(q-1)b}\int \frac{x^{p-1}\,dx}{(a+b\cos x)^{q-1}}.$

**3.** $\displaystyle\int \frac{x\,dx}{1+\sin x} = -x\operatorname{tg}\frac{\pi-2x}{4} + 2\ln\left|\cos\frac{\pi-2x}{4}\right|.$

**4.** $\displaystyle\int \frac{x\,dx}{1-\sin x} = x\operatorname{ctg}\frac{\pi-2x}{4} + 2\ln\left|\sin\frac{\pi-2x}{4}\right|.$

**5.** $\displaystyle\int \frac{x\,dx}{1+\cos x} = x\operatorname{tg}\frac{x}{2} + 2\ln\left|\cos\frac{x}{2}\right|.$

**6.** $\displaystyle\int \frac{x\,dx}{1-\cos x} = -x\operatorname{ctg}\frac{x}{2} + 2\ln\left|\sin\frac{x}{2}\right|.$

**7.** $\displaystyle\int \frac{x\cos x\,dx}{(1+\sin x)^2} = -\frac{x}{1+\sin x} + \operatorname{tg}\frac{2x-\pi}{4}.$

**8.** $\displaystyle\int \frac{x\cos x\,dx}{(1-\sin x)^2} = \frac{x}{1-\sin x} + \operatorname{ctg}\frac{2x-\pi}{4}.$

**9.** $\displaystyle\int \frac{x\sin x\,dx}{(1+\cos x)^2} = \frac{x}{1+\cos x} - \operatorname{tg}\frac{x}{2}.$

**10.** $\displaystyle\int \frac{x\sin x\,dx}{(1-\cos x)^2} = -\frac{x}{1-\cos x} - \operatorname{ctg}\frac{x}{2}.$

**11.** $\displaystyle\int \frac{x\,dx}{(\cos x + a\sin x)^2} = \frac{1}{a^2+1}\ln|\cos x + a\sin x| + \frac{x(\operatorname{tg} x - a)}{(a^2+1)(a\operatorname{tg} x + 1)}.$

**12.** $\displaystyle\int \frac{x^2\,dx}{[(ax-b)\sin x+(a+bx)\cos x]^2} = \frac{x\sin x+\cos x}{b[(ax-b)\sin x+(a+bx)\cos x]}.$

**13.** $\displaystyle\int \frac{\cos^2 x\,dx}{[a\cos x+(ax+b)\sin x]^2} = \frac{\operatorname{tg} x}{a[a+(ax+b)\operatorname{tg} x]}.$

### 5.58. Интегралы вида $\displaystyle\int \frac{x\sin^m x\cos^n x}{(1-k^2\sin^2 x)^r}\,dx$.

Обозначение: $\Delta=\sqrt{1-k^2\sin^2 x}$, $0<k^2<1$.

**1.** $\displaystyle\int \frac{x\sin x\,dx}{\Delta^3} = -\frac{x\cos x}{(1-k^2)\Delta} + \frac{1}{k(1-2k^2)}\arcsin(k\sin x).$

**2.** $\displaystyle\int \frac{x\cos x\,dx}{\Delta^3} = \frac{x\sin x}{\Delta} + \frac{1}{k}\ln(k\cos x+\Delta).$

### 5.59. Интегралы вида $\displaystyle\int (x+b)^{\pm n}\sin ax\,dx$.

**1.** $$\int (x+b)^n\sin ax\,dx = \frac{n!}{a}\left[\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k(x+b)^{n-2k-1}}{(n-2k-1)!\,a^{2k+1}}\sin ax - \right.$$
$$\left.-\sum_{k=0}^{[n/2]}\frac{(-1)^k(x+b)^{n-2k}}{(n-2k)!\,a^{2k}}\cos ax\right].$$

**2.** $\displaystyle\int (x+b)\sin ax\,dx = \frac{1}{a^2}\sin ax - \frac{x+b}{a}\cos ax.$

**3.** $\displaystyle\int (x+b)^2\sin ax\,dx = \frac{2(x+b)}{a^2}\sin ax - \frac{a^2(x+b)^2-2}{a^3}\cos ax.$

**4.** $\displaystyle\int (x+b)^3\sin ax\,dx = \frac{3[a^2(x+b)^2-2]}{a^4}\sin ax - \frac{x+b}{a^3}[a^2(x+b)^2-6]\cos ax.$

**5.** $\displaystyle\int \frac{\sin ax}{(x+b)^n}\,dx = a^{n-1}\left(\cos ab\int\frac{\sin t}{t^n}\,dt - \sin ab\int\frac{\cos t}{t^n}\,dt\right) \qquad [t=a(x+b)].$

**6.** $\displaystyle\int \frac{\sin ax}{x+b}\,dx = \cos ab\,\mathrm{si}\,(ax+ab) - \sin ab\,\mathrm{ci}\,(ax+ab).$

**7.** $\displaystyle\int \frac{\sin ax}{(x+b)^2}\,dx = -\frac{\sin ax}{x+b} + a\int\frac{\cos ax}{x+b}\,dx.$

**8.** $\displaystyle\int \frac{\sin ax}{(x+b)^3}\,dx = -\frac{\sin ax}{2(x+b)^2} - \frac{a\cos ax}{2(x+b)} - \frac{a^2}{2}\int\frac{\sin ax}{x+b}\,dx.$

### 5.60. Интегралы вида $\displaystyle\int (x+b)^{\pm n}\cos ax\,dx$.

**1.** $$\int (x+b)^n\cos ax\,dx = \frac{n!}{a}\left[\sum_{k=0}^{[n/2]}\frac{(-1)^k(x+b)^{n-2k}}{(n-2k)!\,a^{2k}}\sin ax + \right.$$
$$\left.+\sum_{k=0}^{[(n-1)/2]}\frac{(-1)^k(x+b)^{n-2k-1}}{(n-2k-1)!\,a^{2k+1}}\cos ax\right].$$

**2.** $\displaystyle\int (x+b)\cos ax\,dx = \frac{x+b}{a}\sin ax + \frac{1}{a^2}\cos ax.$

**3.** $\int (x+b)^2 \cos ax\, dx = \frac{a^2(x+b)^2-2}{a^3}\sin ax + \frac{2(x+b)}{a^2}\cos ax.$

**4.** $\int (x+b)^3 \cos ax\, dx = \frac{x+b}{a^3}\,[a^2(x+b)^2-6]\sin ax + \frac{3[a^2(x+b)^2-2]}{a^4}\cos ax.$

**5.** $\int \frac{\cos ax}{(x+b)^n}\, dx = a^{n-1}\left(\sin ab \int \frac{\sin t}{t^n}\, dt + \cos ab \int \frac{\cos t}{t^n}\, dt\right)$

$[t = a(x+b)].$

**6.** $\int \frac{\cos ax}{x+b}\, dx = \sin ab\, \operatorname{si}(ax+ab) + \cos ab\, \operatorname{ci}(ax+ab).$

**7.** $\int \frac{\cos ax}{(x+b)^2}\, dx = -\frac{\cos ax}{x+b} - a\int \frac{\sin ax}{x+b}\, dx.$

**8.** $\int \frac{\cos ax}{(x+b)^3}\, dx = -\frac{\cos ax}{2(x+b)^2} + \frac{a\sin ax}{2(x+b)} - \frac{a^2}{2}\int \frac{\cos ax}{x+b}\, dx.$

### 5.61. Интегралы вида $\int e^{ax}\sin^p bx\, dx$.

**1.** $\int e^{ax}\sin^p bx\, dx = \frac{ae^{ax}}{a^2+b^2p^2}\sin^p bx - \frac{bpe^{ax}}{a^2+b^2p^2}\sin^{p-1} bx\cos bx +$

$+ \frac{b^2p(p-1)}{a^2+b^2p^2}\int e^{ax}\sin^{p-2} bx\, dx.$

**2.** $\int e^{ax}\sin^{2n} bx\, dx = \frac{(2n)!b^{2n}e^{ax}}{[a^2+(2n)^2b^2][a^2+(2n-2)^2b^2]\ldots[a^2+4b^2]a} +$

$+ \sum_{k=0}^{n-1} \frac{(2n)!b^{2k}e^{ax}}{(2n-2k)![a^2+(2n)^2b^2][a^2+(2n-2)^2b^2]\ldots[a^2+(2n-2k)^2b^2]} \times$

$\times [a\sin^{2n-2k} bx - (2n-2k)b\sin^{2n-2k-1} bx\cos bx].$

**3.** $= C_{2n}^{n}\frac{e^{ax}}{2^{2n}a} +$

$+ \frac{e^{ax}}{2^{2n-1}}\sum_{k=1}^{n}\frac{(-1)^k}{a^2+4b^2k^2}C_{2n}^{n-k}(a\cos 2bkx + 2bk\sin 2bkx).$

**4.** $\int e^{ax}\sin^{2n+1} bx\, dx =$

$= \sum_{k=0}^{n}\frac{(2n+1)!e^{ax}}{[a^2+(2n+1)^2b^2][a^2+(2n-1)^2b^2]\ldots[a^2+(2n-2k+1)^2b^2]} \times$

$\times \frac{b^{2k}}{(2n-2k+1)!}[a\sin^{2n-2k+1} bx - (2n-2k+1)b\sin^{2n-2k} bx\cos bx].$

**5.** $= \frac{e^{ax}}{2^{2n}}\sum_{k=0}^{n}\frac{(-1)^k}{a^2+(2k+1)^2b^2}C_{2n+1}^{n-k} \times$

$\times [a\sin(2k+1)bx - (2k+1)b\cos(2k+1)bx].$

**6.** $\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx).$

**7.** $\int e^{ax}\sin^2 bx\, dx = \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right).$

**5.62.** Интегралы вида $\int e^{ax}\cos^p bx\,dx$.

**1.** $$\int e^{ax}\cos^p bx\,dx = \frac{ae^{ax}}{a^2+b^2p^2}\cos^p bx + \frac{bpe^{ax}}{a^2+b^2p^2}\sin bx\cos^{p-1} bx + \\ + \frac{b^2p(p-1)}{a^2+b^2p^2}\int e^{ax}\cos^{p-2} bx\,dx.$$

**2.** $$\int e^{ax}\cos^{2n} bx\,dx = \frac{(2n)!b^{2n}e^{ax}}{[a^2+(2n)^2b^2][a^2+(2n-2)^2b^2]\ldots[a^2+4b^2]a} + \\ + \sum_{k=0}^{n-1}\frac{(2n)!b^{2k}e^{ax}}{(2n-2k)![a^2+(2n)^2b^2][a^2+(2n-2)^2b^2]\ldots[a^2+(2n-2k)^2b^2]}\times \\ \times[a\cos^{2n-2k} bx + (2n-2k)b\sin bx\cos^{2n-2k-1} bx].$$

**3.** $$= C_{2n}^n\frac{e^{ax}}{2^{2n}a} + \frac{e^{ax}}{2^{2n-1}}\sum_{k=1}^{n}\frac{C_{2n}^{n-k}}{a^2+4b^2k^2}(a\cos 2bkx + 2bk\sin 2bkx).$$

**4.** $$\int e^{ax}\cos^{2n+1} bx\,dx = \\ = \sum_{k=0}^{n}\frac{(2n+1)!e^{ax}}{[a^2+(2n+1)^2b^2][a^2+(2n-1)^2b^2]\ldots[a^2+(2n-2k+1)^2b^2]}\times \\ \times\frac{b^{2k}}{(2n-2k+1)!}[a\cos^{2n-2k+1} bx + (2n-2k+1)b\sin bx\cos^{2n-2k} bx].$$

**5.** $$= \frac{e^{ax}}{2^{2n}}\sum_{k=0}^{n}\frac{C_{2n+1}^{n-k}}{a^2+(2k+1)^2b^2}\times \\ \times[a\cos(2k+1)bx + (2k+1)b\sin(2k+1)bx].$$

**6.** $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$$

**7.** $$\int e^{ax}\cos^2 bx\,dx = \frac{e^{ax}}{2a} + \frac{e^{ax}}{a^2+4b^2}\left(\frac{a}{2}\cos 2bx + b\sin 2bx\right).$$

**5.63.** Интегралы вида $\int e^{ax}\sin^p bx\cos^q cx\,dx$.

**1.** $$\int e^{ax}\sin^p x\cos^q x\,dx = \\ = \frac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^p x\cos^{q-1} x[a\cos x + (p+q)\sin x] - \\ - pa\int e^{ax}\sin^{p-1} x\cos^{q-1} x\,dx + (q-1)(p+q)\int e^{ax}\sin^p x\cos^{q-2} x\,dx\Big\}.$$

**2.** $$= \frac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1} x\cos^q x[a\sin x - (p+q)\cos x] + \\ + qa\int e^{ax}\sin^{p-1} x\cos^{q-1} x\,dx + (p-1)(p+q)\int e^{ax}\sin^{p-2} x\cos^q x\,dx\Big\}.$$

**3.** $$= \frac{1}{a^2+(p+q)^2}\Big\{e^{ax}\sin^{p-1}x\cos^{q-1}x(a\sin x\cos x + q\sin^2 x - p\cos^2 x) + q(q-1)\int e^{ax}\sin^p x\cos^{q-2}x\,dx + p(p-1)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\Big\}.$$

**4.** $$= \frac{1}{a^2+(p+q)^2}\times \Big\{e^{ax}\sin^{p-1}x\cos^{q-1}x(a\sin x\cos x + q\sin^2 x - p\cos^2 x) + q(q-1)\times \int e^{ax}\sin^{p-2}x\cos^{q-2}x\,dx - (q-p)(p+q-1)\int e^{ax}\sin^{p-2}x\cos^q x\,dx\Big\}.$$

**5.** $$= \frac{1}{a^2+(p+q)^2}\times \Big\{e^{ax}\sin^{p-1}x\cos^{q-1}x(a\sin x\cos x + q\sin^2 x - p\cos^2 x) + p(p-1)\times \int e^{ax}\sin^{p-2}x\cos^{q-2}x\,dx + (q-p)(p+q-1)\int e^{ax}\sin^{p}x\cos^{q-2} x\,dx\Big\}.$$

**6.** $$\int e^{ax}\sin bx\cos cx\,dx = \frac{e^{ax}}{2}\left[\frac{a\sin(b+c)x-(b+c)\cos(b+c)x}{a^2+(b+c)^2} + \frac{a\sin(b-c)x-(b-c)\cos(b-c)x}{a^2+(b-c)^2}\right].$$

**7.** $$\int e^{ax}\sin^2 bx\cos cx\,dx = \frac{e^{ax}}{4}\left[2\frac{a\cos cx + c\sin cx}{a^2+c^2} - \frac{a\cos(2b+c)x+(2b+c)\sin(2b+c)x}{a^2+(2b+c)^2} - \frac{a\cos(2b-c)x+(2b-c)\sin(2b-c)x}{a^2+(2b-c)^2}\right].$$

**8.** $$\int e^{ax}\sin bx\cos^2 cx\,dx = \frac{e^{ax}}{4}\left[2\frac{a\sin bx - b\cos bx}{a^2+b^2} + \frac{a\sin(b+2c)x-(b+2c)\cos(b+2c)x}{a^2+(b+2c)^2} + \frac{a\sin(b-2c)x-(b-2c)\cos(b-2c)x}{a^2+(b-2c)^2}\right].$$

**5.64. Интегралы вида** $\int e^{ax}\operatorname{tg}^p x\,dx$.

**1.** $$\int e^{ax}\operatorname{tg}^p x\,dx = \frac{e^{ax}}{p-1}\operatorname{tg}^{p-1}x - \frac{a}{p-1}\int e^{ax}\operatorname{tg}^{p-1}x\,dx - \int e^{ax}\operatorname{tg}^{p-2}x\,dx.$$

**2.** $$\int e^{ax}\operatorname{tg} x\,dx = \frac{e^{ax}}{a}\operatorname{tg} x - \frac{1}{a}\int\frac{e^{ax}\,dx}{\cos^2 x}.$$

**3.** $$\int e^{ax}\operatorname{tg}^2 x\,dx = -\frac{e^{ax}}{a} + e^{ax}\operatorname{tg} x - a\int e^{ax}\operatorname{tg} x\,dx.$$

**5.65.** Интегралы вида $\int e^{ax}\operatorname{ctg}^p x\,dx$.

**1.** $\displaystyle\int e^{ax}\operatorname{ctg}^p x\,dx =$
$$= -\frac{e^{ax}}{p-1}\operatorname{ctg}^{p-1} x + \frac{a}{p-1}\int e^{ax}\operatorname{ctg}^{p-1} x\,dx - \int e^{ax}\operatorname{ctg}^{p-2} x\,dx.$$

**2.** $\displaystyle\int e^{ax}\operatorname{ctg} x\,dx = \frac{e^{ax}}{a}\operatorname{ctg} x + \frac{1}{a}\int \frac{e^{ax}\,dx}{\sin^2 x}.$

**3.** $\displaystyle\int e^{ax}\operatorname{ctg}^2 x\,dx = -\frac{e^{ax}}{a} - e^{ax}\operatorname{ctg} x + a\int e^{ax}\operatorname{ctg} x\,dx.$

**5.66.** Интегралы вида $\int x^p e^{ax}\sin(bx+c)\,dx$.

Обозначение: $\sin\varphi = -\dfrac{b}{\sqrt{a^2+b^2}}$, $\cos\varphi = \dfrac{a}{\sqrt{a^2+b^2}}$.

**1.** $\displaystyle\int x^p e^{ax}\sin bx\,dx = \frac{x^p e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) -$
$$- \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}(a\sin bx - b\cos bx)\,dx.$$

**2.** $$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}}\sin(bx+\varphi) - \frac{p}{\sqrt{a^2+b^2}}\int x^{p-1}e^{ax}\sin(bx+\varphi)\,dx.$$

**3.** $\displaystyle\int x^n e^{ax}\sin bx\,dx = n!\,e^{ax}\sum_{k=1}^{n+1}\frac{(-1)^{k+1}x^{n-k+1}}{(n-k+1)!(a^2+b^2)^{k/2}}\sin(bx+k\varphi).$

**4.** $\displaystyle\int xe^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right)\sin bx - \left(bx - \frac{2ab}{a^2+b^2}\right)\cos bx\right].$

**5.** $\displaystyle\int x^2 e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[ax^2 - \frac{2(a^2-b^2)}{a^2+b^2}x + \frac{2a(a^2-3b^2)}{(a^2+b^2)^2}\right]\sin bx -$
$$- \frac{e^{ax}}{a^2+b^2}\left[bx^2 - \frac{4ab}{a^2+b^2}x + \frac{2b(3a^2-b^2)}{(a^2+b^2)^2}\right]\cos bx.$$

**6.** $\displaystyle\int x^n e^{ax}\sin(bx+c)\,dx =$
$$= \frac{e^{ax}}{n+1}\sum_{k=1}^{n+1}(-1)^{k+1}\frac{(n+1)n\ldots(n-k+2)}{(a^2+b^2)^{k/2}}x^{n-k+1}\sin(bx+c+k\varphi).$$

**7.** $\displaystyle\int xe^{ax}\sin(bx+c)\,dx =$
$$= \frac{xe^{ax}}{\sqrt{a^2+b^2}}\sin(bx+c+\varphi) - \frac{e^{ax}}{a^2+b^2}\sin(bx+c+2\varphi).$$

**5.67.** Интегралы вида $\int x^p e^{ax}\cos(bx+c)\,dx$.

Обозначение: $\sin\varphi = -\dfrac{b}{\sqrt{a^2+b^2}}$, $\cos\varphi = \dfrac{a}{\sqrt{a^2+b^2}}$.

**1.** $\displaystyle\int x^p e^{ax}\cos bx\,dx = \frac{x^p e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) -$
$$- \frac{p}{a^2+b^2}\int x^{p-1}e^{ax}(a\cos bx + b\sin bx)\,dx.$$

**2.** $$= \frac{x^p e^{ax}}{\sqrt{a^2+b^2}} \cos(bx+\varphi) - \frac{p}{\sqrt{a^2+b^2}} \int x^{p-1} e^{ax} \cos(bx+\varphi)\, dx.$$

**3.** $$\int x^n e^{ax} \cos bx\, dx = n!\, e^{ax} \sum_{k=1}^{n+1} \frac{(-1)^{k+1} x^{n-k+1}}{(n-k+1)!(a^2+b^2)^{k/2}} \cos(bx+k\varphi).$$

**4.** $$\int x e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[\left(ax - \frac{a^2-b^2}{a^2+b^2}\right)\cos bx + \left(bx - \frac{2ab}{a^2+b^2}\right)\sin bx\right].$$

**5.** $$\int x^2 e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[ax^2 - \frac{2(a^2-b^2)}{a^2+b^2}\, x + \frac{2a(a^2-3b^2)}{(a^2+b^2)^2}\right]\cos bx + \\ + \frac{e^{ax}}{a^2+b^2}\left[bx^2 - \frac{4ab}{a^2+b^2}\, x + \frac{2b(3a^2-b^2)}{(a^2+b^2)^2}\right]\sin bx.$$

**6.** $$\int x^n e^{ax} \cos(bx+c)\, dx = \\ = \frac{e^{ax}}{n+1} \sum_{k=1}^{n+1} (-1)^{k+1} \frac{(n+1)n\ldots(n-k+2)}{(a^2+b^2)^{k/2}}\, x^{n-k+1} \cos(bx+c+k\varphi).$$

**7.** $$\int x e^{ax} \cos(bx+c)\, dx = \\ = \frac{x e^{ax}}{\sqrt{a^2+b^2}} \cos(bx+c+\varphi) - \frac{e^{ax}}{a^2+b^2} \cos(bx+c+2\varphi).$$

## **5.68.** Интегралы вида $\int \operatorname{sh}^m(ax+b) \sin^n(cx+d)\, dx$.

**1.** $$\int \operatorname{sh}(ax+b) \sin(cx+d)\, dx = \frac{a}{a^2+c^2} \operatorname{ch}(ax+b) \sin(cx+d) - \\ - \frac{c}{a^2+b^2} \operatorname{sh}(ax+b) \cos(cx+d).$$

**2.** $$\int \operatorname{sh} x \sin x\, dx = \frac{1}{2} \operatorname{ch} x \sin x - \frac{1}{2} \operatorname{sh} x \cos x.$$

**3.** $$\int \operatorname{sh}^{2m}(ax+b) \sin^{2n}(cx+d)\, dx = \frac{(-1)^m}{2^{2m+2n}} C_{2m}^m C_{2n}^n x + \\ + \frac{1}{2^{2m+2n-1}} C_{2n}^n \sum_{j=0}^{m-1} \frac{(-1)^j}{(2m-2j)a} C_{2m}^j \operatorname{sh}[(2m-2j)(ax+b)] + \\ + \frac{(-1)^{m+n}}{2^{2m+2n-1}} C_{2m}^m \sum_{k=0}^{n-1} \frac{(-1)^k}{(2n-2k)c} C_{2n}^k \sin[(2n-2k)(cx+d)] + \\ + \frac{(-1)^n}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{(-1)^{j+k} C_{2m}^j C_{2n}^k}{(2m-2j)^2 a^2 + (2n-2k)^2 c^2} \times \\ \times \{(2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]\}.$$

**4.** $$\int \operatorname{sh}^{2m}(ax+b) \sin^{2n+1}(cx+d)\, dx = \\ = \frac{(-1)^{m+n+1}}{2^{2m+2n}} C_{2m}^m \sum_{k=0}^{n} \frac{(-1)^k}{(2n-2k+1)c} C_{2n+1}^k \cos[(2n-2k+1)(cx+d)] +$$

$$+\frac{(-1)^n}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\sum_{k=0}^{n}\frac{(-1)^{j+k}C_{2m}^{j}C_{2n+1}^{k}}{(2m-2j)^2a^2+(2n-2k+1)^2c^2}\times$$
$$\times\{(2m-2j)a\,\mathrm{sh}\,[(2m-2j)(ax+b)]\sin[(2n-2k+1)(cx+d)]-$$
$$-(2n-2k+1)c\,\mathrm{ch}\,[(2m-2j)(ax+b)]\cos[(2n-2k+1)(cx+d)]\}.$$

**5.** $\int \mathrm{sh}^{2m+1}(ax+b)\sin^{2n}(cx+d)\,dx=$

$$=\frac{C_{2n}^{n}}{2^{2m+2n}}\sum_{j=0}^{m}\frac{(-1)^jC_{2m+1}^{j}}{(2m-2j+1)a}\,\mathrm{ch}\,[(2m-2j+1)(ax+b)]+$$
$$+\frac{(-1)^n}{2^{2m+2n-1}}\sum_{j=0}^{m}\sum_{k=0}^{n-1}\frac{(-1)^{j+k}C_{2m+1}^{j}C_{2n}^{k}}{(2m-2j+1)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j+1)a\,\mathrm{ch}\,[(2m-2j+1)(ax+b)]\cos[(2n-2k)(cx+d)]+$$
$$+(2n-2k)c\,\mathrm{sh}\,[(2m-2j+1)(ax+b)]\sin[(2n-2k)(cx+d)]\}.$$

**6.** $\int \mathrm{sh}^{2m+1}(ax+b)\sin^{2n+1}(cx+d)\,dx=$

$$=\frac{(-1)^n}{2^{2m+2n}}\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(-1)^{j+k}C_{2m+1}^{j}C_{2n+1}^{k}}{(2m-2j+1)^2a^2+(2n-2k+1)^2c^2}\times$$
$$\times\{(2m-2j+1)a\,\mathrm{ch}\,[(2m-2j+1)(ax+b)]\sin[(2n-2k+1)(cx+d)]-$$
$$-(2n-2k+1)c\,\mathrm{sh}\,[(2m-2j+1)(ax+b)]\cos[(2n-2k+1)(cx+d)]\}.$$

### 5.69. Интегралы вида $\int \mathrm{sh}^m(ax+b)\cos^n(cx+d)\,dx$.

**1.** $$\int \mathrm{sh}\,(ax+b)\cos(cx+d)\,dx=\frac{a}{a^2+c^2}\,\mathrm{ch}\,(ax+b)\cos(cx+d)+$$
$$+\frac{c}{a^2+c^2}\,\mathrm{sh}\,(ax+b)\sin(cx+d).$$

**2.** $\int \mathrm{sh}\,x\cos x\,dx=\frac{1}{2}\,\mathrm{ch}\,x\cos x+\frac{1}{2}\,\mathrm{sh}\,x\sin x.$

**3.** $\int \mathrm{sh}^{2m}(ax+b)\cos^{2n}(cx+d)\,dx=$

$$=\frac{(-1)^m}{2^{2m+2n}}C_{2m}^{m}C_{2n}^{n}x+\frac{C_{2n}^{n}}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\frac{(-1)^jC_{2m}^{j}}{(2m-2j)a}\times$$
$$\times\mathrm{sh}\,[(2m-2j)(ax+b)]+\frac{(-1)^mC_{2m}^{m}}{2^{2m+2n-1}}\sum_{k=0}^{n-1}\frac{C_{2n}^{k}}{(2n-2k)c}\sin[(2n-2k)(cx+d)]+$$
$$+\frac{1}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^jC_{2m}^{j}C_{2n}^{k}}{(2m-2j)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j)a\,\mathrm{sh}\,[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]+$$
$$+(2n-2k)c\,\mathrm{ch}\,[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]\}.$$

**4.** $\int \mathrm{sh}^{2m}(ax+b)\cos^{2n+1}(cx+d)\,dx=$

$$=\frac{(-1)^mC_{2m}^{m}}{2^{2m+2n}}\sum_{k=0}^{n}\frac{C_{2n+1}^{k}}{(2n-2k+1)c}\sin[(2n-2k+1)(cx+d)]+$$

$$+\frac{1}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\sum_{k=0}^{n}\frac{(-1)^j C_{2m}^j C_{2n+1}^k}{(2m-2j)^2a^2+(2n-2k+1)^2c^2}\times$$
$$\times\{(2m-2j)a\operatorname{sh}[(2m-2j)(ax+b)]\cos[(2n-2k+1)(cx+d)]+$$
$$+(2n-2k+1)c\operatorname{ch}[(2m-2j)(ax+b)]\sin[(2n-2k+1)(cx+d)]\}.$$

**5.** $\displaystyle\int \operatorname{sh}^{2m+1}(ax+b)\cos^{2n}(cx+d)\,dx =$

$$=\frac{C_{2n}^n}{2^{2m+2n}}\sum_{j=0}^{m}\frac{(-1)^j C_{2m+1}^j}{(2m-2j+1)a}\operatorname{ch}[(2m-2j+1)(ax+b)]+$$
$$+\frac{1}{2^{2m+2n-1}}\sum_{j=0}^{m}\sum_{k=0}^{n-1}\frac{(-1)^j C_{2m+1}^j C_{2n}^k}{(2m-2j+1)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j+1)a\operatorname{ch}[(2m-2j+1)(ax+b)]\cos[(2n-2k)(cx+d)]+$$
$$+(2n-2k)c\operatorname{sh}[(2m-2j+1)(ax+b)]\sin[(2n-2k)(cx+d)]\}.$$

**6.** $\displaystyle\int \operatorname{sh}^{2m+1}(ax+b)\cos^{2n+1}(cx+d)\,dx =$

$$=\frac{1}{2^{2m+2n}}\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{(-1)^j C_{2m+1}^j C_{2n+1}^k}{(2m-2j+1)^2a^2+(2n-2k+1)^2c^2}\times$$
$$\times\{(2m-2j+1)a\operatorname{ch}[(2m-2j+1)(ax+b)]\cos[(2n-2k+1)(cx+d)]+$$
$$+(2n-2k+1)c\operatorname{sh}[(2m-2j+1)(ax+b)]\sin[(2n-2k+1)(cx+d)]\}.$$

### 5.70. Интегралы вида $\int \operatorname{ch}^m(ax+b)\sin^n(cx+d)\,dx$.

**1.** $\displaystyle\int \operatorname{ch}(ax+b)\sin(cx+d)\,dx = \frac{a}{a^2+c^2}\operatorname{sh}(ax+b)\sin(cx+d) -$
$$-\frac{c}{a^2+c^2}\operatorname{ch}(ax+b)\cos(cx+d).$$

**2.** $\displaystyle\int \operatorname{ch}x\sin x\,dx = \frac{1}{2}\operatorname{sh}x\sin x - \frac{1}{2}\operatorname{ch}x\cos x.$

**3.** $\displaystyle\int \operatorname{ch}^{2m}(ax+b)\sin^{2n}(cx+d)\,dx = \frac{C_{2m}^m C_{2n}^n}{2^{2m+2n}}x + \frac{(-1)^n C_{2m}^m}{2^{2m+2n-1}}\sum_{k=0}^{m-1}\frac{(-1)^k C_{2n}^k}{(2n-2k)c}\times$

$$\times\sin[(2n-2k)(cx+d)]+\frac{C_{2n}^n}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\frac{C_{2m}^j}{(2m-2j)a}\operatorname{sh}[(2m-2j)(ax+b)]+$$
$$+\frac{(-1)^n}{2^{2m+2n-2}}\sum_{j=0}^{m-1}\sum_{k=0}^{n-1}\frac{(-1)^k C_{2m}^j C_{2n}^k}{(2m-2j)^2a^2+(2n-2k)^2c^2}\times$$
$$\times\{(2m-2j)a\operatorname{sh}[(2m-2j)(ax+b)]\cos[(2n-2k)(cx+d)]+$$
$$+(2n-2k)c\operatorname{ch}[(2m-2j)(ax+b)]\sin[(2n-2k)(cx+d)]\}.$$

**4.** $\displaystyle\int \operatorname{ch}^{2m+1}(ax+b)\sin^{2n}(cx+d)\,dx =$

$$=\frac{C_{2n}^n}{2^{2m+2n}}\sum_{j=0}^{m}\frac{C_{2m+1}^j}{(2m-2j+1)a}\operatorname{sh}[(2m-2j+1)(ax+b)]+$$
$$+\frac{(-1)^n}{2^{2m+2n-1}}\sum_{j=0}^{m}\sum_{k=0}^{n-1}\frac{(-1)^k C_{2m+1}^j C_{2n}^k}{(2m-2j+1)^2a^2+(2n-2k)^2c^2}\times$$

$$\times \{(2m-2j+1)a \operatorname{sh}[(2m-2j+1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{ch}[(2m-2j+1)(ax+b)] \sin[(2n-2k)(cx+d)]\}.$$

**5.** $\displaystyle\int \operatorname{ch}^{2m}(ax+b) \sin^{2n+1}(cx+d)\,dx =$

$$= \frac{(-1)^n C_{2m}^m}{2^{2m+2n}} \sum_{k=0}^{n} \frac{(-1)^{k+1} C_{2n+1}^k}{(2n-2k+1)c} \cos[(2n-2k+1)(cx+d)] + \\ + \frac{(-1)^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \sum_{k=0}^{n} \frac{(-1)^k C_{2m}^j C_{2n+1}^k}{(2m-2j)^2a^2 + (2n-2k+1)^2c^2} \times \\ \times \{(2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \sin[(2n-2k+1)(cx+d)] - \\ - (2n-2k+1)c \operatorname{ch}[(2m-2j)(ax+b)] \cos[(2n-2k+1)(cx+d)]\}.$$

**6.** $\displaystyle\int \operatorname{ch}^{2m+1}(ax+b) \sin^{2n+1}(cx+d)\,dx =$

$$= \frac{(-1)^n}{2^{2m+2n}} \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{(-1)^k C_{2m+1}^j C_{2n+1}^k}{(2m-2j+1)^2a^2 + (2n-2k+1)^2c^2} \times \\ \times \{(2m-2j+1)a \operatorname{sh}[(2m-2j+1)(ax+b)] \sin[(2n-2k+1)(cx+d)] - \\ - (2n-2k+1)c \operatorname{ch}[(2m-2j+1)(ax+b)] \cos[(2n-2k+1)(cx+d)]\}.$$

### 5.71. Интегралы вида $\int \operatorname{ch}^m(ax+b) \cos^n(cx+d)\,dx$.

**1.** $$\int \operatorname{ch}(ax+b) \cos(cx+d)\,dx = \frac{a}{a^2+c^2} \operatorname{sh}(ax+b) \cos(cx+d) + \\ + \frac{c}{a^2+c^2} \operatorname{ch}(ax+b) \sin(cx+d).$$

**2.** $\displaystyle\int \operatorname{ch} x \cos x\,dx = \frac{1}{2} \operatorname{sh} x \cos x + \frac{1}{2} \operatorname{ch} x \sin x.$

**3.** $$\int \operatorname{ch}^{2m}(ax+b) \cos^{2n}(cx+d)\,dx = \frac{C_{2m}^m C_{2n}^n}{2^{2m+2n}}\, x + \frac{C_{2m}^m}{2^{2m+2n-1}} \sum_{k=0}^{n-1} \frac{C_{2n}^k}{(2n-2k)c} \times \\ \times \sin[(2n-2k)(cx+d)] + \frac{C_{2n}^n}{2^{2m+2n-1}} \sum_{j=0}^{m-1} \frac{C_{2m}^j}{(2m-2j)a} \operatorname{sh}[(2m-2j)(ax+b)] + \\ + \frac{1}{2^{2m+2n-2}} \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{C_{2m}^j C_{2n}^k}{(2m-2j)^2a^2 + (2n-2k)^2c^2} \times \\ \times \{(2m-2j)a \operatorname{sh}[(2m-2j)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{ch}[(2m-2j)(ax+b)] \sin[(2n-2k)(cx+d)]\}.$$

**4.** $\displaystyle\int \operatorname{ch}^{2m+1}(ax+b) \cos^{2n}(cx+d)\,dx =$

$$= \frac{C_{2n}^n}{2^{2m+2n}} \sum_{j=0}^{m} \frac{C_{2m+1}^j}{(2m-2j+1)a} \operatorname{sh}[(2m-2j+1)(ax+b)] + \\ + \frac{1}{2^{2m+2n-1}} \sum_{j=0}^{m} \sum_{k=0}^{n-1} \frac{C_{2m+1}^j C_{2n}^k}{(2m-2j+1)^2a^2 + (2n-2k)^2c^2} \times \\ \times \{(2m-2j+1)a \operatorname{sh}[(2m-2j+1)(ax+b)] \cos[(2n-2k)(cx+d)] + \\ + (2n-2k)c \operatorname{ch}[(2m-2j+1)(ax+b)] \sin[(2n-2k)(cx+d)]\}.$$

**5.** $\displaystyle\int \operatorname{ch}^{2m}(ax+b)\cos^{2n+1}(cx+d)\,dx =$

$$= \frac{C_{2m}^{m}}{2^{2m+2n}}\sum_{k=0}^{n}\frac{C_{2n+1}^{k}}{(2n-2k+1)c}\sin\left[(2n-2k+1)(cx+d)\right] +$$

$$+\frac{1}{2^{2m+2n-1}}\sum_{j=0}^{m-1}\sum_{k=0}^{n}\frac{C_{2m}^{j}C_{2n+1}^{k}}{(2m-2j)^2a^2+(2n-2k+1)^2c^2}\times$$

$$\times\{(2m-2j)a\operatorname{sh}[(2m-2j)(ax+b)]\cos[(2n-2k+1)(cx+d)] +$$
$$+(2n-2k+1)c\operatorname{ch}[(2m-2j)(ax+b)]\sin[(2n-2k+1)(cx+d)]\}.$$

**6.** $\displaystyle\int \operatorname{ch}^{2m+1}(ax+b)\cos^{2n+1}(cx+d)\,dx =$

$$= \frac{1}{2^{2m+2n}}\sum_{j=0}^{m}\sum_{k=0}^{n}\frac{C_{2m+1}^{j}C_{2n+1}^{k}}{(2m-2j+1)^2a^2+(2n-2k+1)^2c^2}\times$$

$$\times\{(2m-2j+1)a\operatorname{sh}[(2m-2j+1)(ax+b)]\cos[(2n-2k+1)(cx+d)] +$$
$$+(2n-2k+1)c\operatorname{ch}[(2m-2j+1)(ax+b)]\sin[(2n-2k+1)(cx+d)]\}.$$

### 5.72. Интегралы вида $\int x^p \sin x^2\,dx$.

Интегралы $\int x^{2n+1}\sin x^2\,dx$ выражаются через элементарные функции, интегралы $\int x^{\pm 2n}\sin x^2\,dx$ — через синус-интеграл $\mathrm{S}(x)$ и косинус-интеграл $\mathrm{C}(x)$ Френеля, а интегралы $\int x^{-2n-1}\sin x^2\,dx$ — через интегральный синус $\operatorname{si}(x)$ и интегральный косинус $\operatorname{ci}(x)$.

**1.** $\displaystyle\int x^p\sin x^2\,dx = -\frac{x^{p-1}}{2}\cos x^2 + \frac{p-1}{p}\int x^{p-2}\cos x^2\,dx.$

**2.** $\displaystyle\int \sin x^2\,dx = \sqrt{\frac{\pi}{2}}\,\mathbf{S}(x^2).$

**3.** $\displaystyle\int x\sin x^2\,dx = -\frac{1}{2}\cos x^2.$

**4.** $\displaystyle\int x^2\sin x^2\,dx = -\frac{x}{2}\cos x^2 + \frac{1}{2}\sqrt{\frac{\pi}{2}}\,\mathbf{C}(x^2).$

**5.** $\displaystyle\int x^3\sin x^2\,dx = -\frac{x^2}{2}\cos x^2 + \frac{1}{2}\sin x^2.$

**6.** $\displaystyle\int \frac{\sin x^2}{x}\,dx = \frac{1}{2}\operatorname{si}(x^2).$

**7.** $\displaystyle\int \frac{\sin x^2}{x^2}\,dx = -\frac{\sin x^2}{x} + \sqrt{2\pi}\,\mathbf{C}(x^2).$

**8.** $\displaystyle\int \sin(ax^2+bx+c)\,dx = \sqrt{\frac{\pi}{2a}}\left[\cos\frac{4ac-b^2}{4a}\mathbf{S}\left(\frac{(2ax+b)^2}{4a}\right) +\right.$

$$\left. +\sin\frac{4ac-b^2}{4a}\mathbf{C}\left(\frac{(2ax+b)^2}{4a}\right)\right].$$

**5.73.** Интегралы вида $\int x^p \cos x^2\,dx$.

Интегралы $\int x^{2n+1}\cos x^2\,dx$ выражаются через элементарные функции, интегралы $\int x^{\pm 2n}\cos x^2\,dx$ — через синус-интеграл $\mathrm{S}(x)$ и косинус-интеграл $\mathrm{C}(x)$ Френеля, а интегралы $\int x^{-2n-1}\cos x^2\,dx$ — через интегральный синус $\operatorname{si}(x)$ и интегральный косинус $\operatorname{ci}(x)$.

**1.** $\displaystyle\int x^p\cos x^2\,dx = \frac{x^{p-1}}{2}\sin x^2 - \frac{p-1}{p}\int x^{p-2}\sin x^2\,dx.$

**2.** $\displaystyle\int \cos x^2\,dx = \sqrt{\frac{\pi}{2}}\,\mathbf{C}(x^2).$

**3.** $\displaystyle\int x\cos x^2\,dx = \frac{1}{2}\sin x^2.$

**4.** $\displaystyle\int x^2\cos x^2\,dx = \frac{x}{2}\sin x^2 - \frac{1}{2}\sqrt{\frac{\pi}{2}}\,\mathbf{S}(x^2).$

**5.** $\displaystyle\int x^3\cos x^2\,dx = \frac{x^2}{2}\sin x^2 + \frac{1}{2}\cos x^2.$

**6.** $\displaystyle\int \frac{\cos x^2}{x}\,dx = \frac{1}{2}\operatorname{ci}(x^2).$

**7.** $\displaystyle\int \frac{\cos x^2}{x^2}\,dx = -\frac{\cos x^2}{x} - \sqrt{2\pi}\,\mathbf{S}(x^2).$

**8.** $$\int \cos(ax^2+bx+c)\,dx = \sqrt{\frac{\pi}{2a}}\left[\cos\frac{4ac-b^2}{4a}\mathbf{C}\left(\frac{(2ax+b)^2}{4a}\right) - \right.$$
$$\left. - \sin\frac{4ac-b^2}{4a}\mathbf{S}\left(\frac{(2ax+b)^2}{4a}\right)\right].$$

## 6. ЛОГАРИФМИЧЕСКАЯ ФУНКЦИЯ

### 6.1. Интегралы вида $\int x^p \ln^q x\, dx$.

1. $\int f(x)\ln g(x)\,dx = F(x)\ln g(x) - \int \frac{F(x)g'(x)}{g(x)}\,dx \qquad \left[F(x) = \int f(x)\,dx\right].$

2. $\int \ln g(x)\,dx = x\ln g(x) - \int \frac{xg'(x)}{g(x)}\,dx.$

3. $\int f(x)\ln^n x\,dx = F(x)\ln^n x - n\int \frac{F(x)}{x}\ln^{n-1} x\,dx \qquad \left[F(x) = \int f(x)\,dx\right].$

4. $\int x^p \ln^q x\,dx = \frac{x^{p+1}\ln^{q+1} x}{q+1} - \frac{p+1}{q+1}\int x^p \ln^{q+1} x\,dx.$

5. $\qquad = \frac{x^{p+1}\ln^q x}{p+1} - \frac{q}{p+1}\int x^p \ln^{q-1} x\,dx.$

6. $\int x^p \ln^n x\,dx = \frac{x^{p+1}}{n+1}\sum_{k=0}^{n}(-1)^k (n+1)n(n-1)\ldots(n-k+1)\frac{\ln^{n-k} x}{(p+1)^{k+1}}.$

7. $\qquad = \frac{\ln^{n+1} x}{n+1} \qquad [p=-1].$

8. $\int \frac{x^p}{\ln x}\,dx = \ln|\ln x| + \sum_{k=1}^{\infty}\frac{(p+1)^k}{k!k}\ln^k x.$

9. $\int \frac{\ln^q x}{x^p}\,dx = -\frac{\ln^q x}{(p-1)x^{p-1}} + \frac{q}{p-1}\int \frac{\ln^{q-1} x}{x^p}\,dx.$

10. $\int \frac{\ln^q x}{x}\,dx = \frac{\ln^{q+1} x}{q+1}.$

11. $\int \frac{\ln^n x}{x^m}\,dx = -\frac{1}{(n+1)x^{m-1}}\sum_{k=0}^{n}\frac{(n+1)n\ldots(n-k+1)}{(m-1)^{k+1}}\ln^{n-k} x.$

12. $\int \ln^q x\,dx = x\ln^q x - q\int \ln^{q-1} x\,dx.$

13. $\int \ln^n x\,dx = x\sum_{k=0}^{n}(-1)^k k!\,C_n^k \ln^{n-k} x.$

14. $\int \ln x\,dx = x\ln x - x.$

15. $\int \ln^2 x\,dx = x\ln^2 x - 2x\ln x + 2x.$

16. $\int \ln^3 x\,dx = x\ln^3 x - 3x\ln^2 x + 6x\ln x - 6x.$

**17.** $\int x^p \ln x\, dx = x^{p+1}\left[\dfrac{\ln x}{p+1} - \dfrac{1}{(p+1)^2}\right].$

**18.** $\int x^p \ln^2 x\, dx = x^{p+1}\left[\dfrac{\ln^2 x}{p+1} - \dfrac{2\ln x}{(p+1)^2} + \dfrac{2}{(p+1)^3}\right].$

**19.** $\int x^p \ln^3 x\, dx = x^{p+1}\left[\dfrac{\ln^3 x}{p+1} - \dfrac{3\ln^2 x}{(p+1)^2} + \dfrac{6\ln x}{(p+1)^3} - \dfrac{6}{(p+1)^4}\right].$

**20.** $\int \dfrac{\ln x}{x}\, dx = \dfrac{\ln^2 x}{2}.$

**21.** $\int \dfrac{\ln x}{x^2}\, dx = -\dfrac{\ln x}{x} - \dfrac{1}{x}.$

**22.** $\int \dfrac{\ln x}{x^3}\, dx = -\dfrac{\ln x}{2x^2} - \dfrac{1}{4x^2}.$

**23.** $\int \dfrac{\ln^2 x}{x}\, dx = \dfrac{\ln^3 x}{3}.$

**24.** $\int \dfrac{\ln^2 x}{x^2}\, dx = -\dfrac{\ln^2 x}{x} - \dfrac{2\ln x}{x} - \dfrac{2}{x}.$

**25.** $\int \dfrac{\ln^2 x}{x^3}\, dx = -\dfrac{\ln^2 x}{2x^2} - \dfrac{\ln x}{2x^2} - \dfrac{1}{4x^2}.$

### 6.2. Интегралы вида $\int \dfrac{x^p\, dx}{\ln^q x}$.

**1.** $\int \dfrac{x^p\, dx}{\ln^q x} = -\dfrac{x^{p+1}}{(q-1)\ln^{q-1} x} + \dfrac{p+1}{q-1}\int \dfrac{x^p\, dx}{\ln^{q-1} x}.$

**2.** $\int \dfrac{dx}{x^p \ln^q x} = -\dfrac{1}{(q-1)x^{p-1}\ln^{q-1} x} - \dfrac{p-1}{q-1}\int \dfrac{dx}{x^p \ln^{q-1} x}.$

**3.** $\int \dfrac{dx}{x^p \ln x} = \ln|\ln x| + \sum\limits_{k=1}^{\infty} (-1)^k \dfrac{(p-1)^k}{k!k} \ln^k x.$

**4.** $\int \dfrac{dx}{x\ln x} = \ln|\ln x|.$

**5.** $\int \dfrac{dx}{x\ln^q x} = -\dfrac{1}{(q-1)\ln^{q-1} x}.$

### 6.3. Интегралы вида $\int (x+a)^p \ln x\, dx$.

**1.** $\int (x+a)^m \ln x\, dx = \dfrac{1}{m+1}(x+a)^{m+1}\ln x - \dfrac{1}{m+1}\int \dfrac{(x+a)^{m+1}}{x}\, dx.$

**2.** $\qquad = \dfrac{1}{m+1}\left[(x+a)^{m+1} - a^{m+1}\right]\ln x - \sum\limits_{k=0}^{m} C_m^k \dfrac{a^{m-k}x^{k+1}}{(k+1)^2}.$

**3.** $\int (x+a)\ln x\, dx = \dfrac{1}{2}(x^2+2ax)\ln x - ax - \dfrac{x^2}{4}.$

**4.** $\int (x+a)^2 \ln x\, dx = \dfrac{1}{3}(x^3+3ax^2+3a^2x)\ln x - \dfrac{x^3}{9} - \dfrac{ax^2}{2} - a^2x.$

**5.** $\int (x+a)^3 \ln x\, dx = \frac{1}{4}\left[(x+a)^4 - a^4\right] \ln x - \frac{x^4}{16} - \frac{ax^3}{3} - \frac{3a^2x^2}{4} - a^3 x.$

**6.** $\int \frac{\ln^n x\, dx}{(x+a)^m} = \frac{x \ln^n x}{(m-1)a(x+a)^{m-1}} - \frac{n}{(m-1)a} \int \frac{\ln^{n-1} x\, dx}{(x+a)^{m-1}} +$

$$+ \frac{m-2}{(m-1)a} \int \frac{\ln^n x\, dx}{(x+a)^{m-1}}.$$

**7.** $\int \frac{\ln x\, dx}{(x+a)^m} == -\frac{\ln x}{(m-1)(x+a)^{m-1}} +$

$$+ \frac{1}{(m-1)a^{m-1}} \left[\ln \frac{x}{x+a} + \sum_{k=1}^{m-2} \frac{(-1)^k}{k} C_{m-2}^k \left(\frac{x}{x+a}\right)^k\right].$$

**8.** $\int \frac{\ln x\, dx}{x+a} = \ln x \cdot \ln \frac{x+a}{a} + \mathrm{Li}_2\left(-\frac{x}{a}\right).$

**9.** $\int \frac{\ln x\, dx}{(x+a)^2} = -\frac{\ln x}{x+a} + \frac{1}{a} \ln \frac{x}{x+a}.$

**10.** $\int \frac{\ln x\, dx}{(x+a)^3} = -\frac{\ln x}{2(x+a)^2} + \frac{1}{2a(x+a)} + \frac{1}{2a^2} \ln \frac{x}{x+a}.$

**11.** $\int \frac{\ln x\, dx}{(x+a)^{1/2}} = 2\left[(\ln x - 2)(x+a)^{1/2} + a^{1/2} \ln \frac{(x+a)^{1/2} + a^{1/2}}{(x+a)^{1/2} - a^{1/2}}\right]$

$[a > 0].$

**12.** $= 2\left[(\ln x - 2)(x+a)^{1/2} + 2(-a)^{1/2} \operatorname{arctg} \left(\frac{x+a}{-a}\right)^{1/2}\right]$ $[a < 0].$

## 6.4. Интегралы вида $\int x^p \ln(ax+b)\, dx$.

**1.** $\int f(x) \ln^n(ax+b)\, dx = \frac{1}{a} \int f\left(\frac{t-b}{a}\right) \ln^n t\, dt$ $[t = ax+b].$

**2.** $\int x^m \ln(ax+b)\, dx = \frac{1}{m+1}\left[x^{m+1} + (-1)^m \frac{b^{m+1}}{a^{m+1}}\right] \ln(ax+b) +$

$$+ \frac{1}{m+1} \sum_{k=1}^{m+1} \frac{(-1)^k x^{m-k+2} b^{k-1}}{(m-k+2)a^{k-1}}.$$

**3.** $\int \ln(ax+b)\, dx = \frac{1}{a}(ax+b) \ln(ax+b) - x.$

**4.** $\int x \ln(ax+b)\, dx = \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right) \ln(ax+b) - \frac{1}{2}\left(\frac{x^2}{2} - \frac{bx}{a}\right).$

**5.** $\int x^2 \ln(ax+b)\, dx = \frac{1}{3}\left(x^3 + \frac{b^3}{a^3}\right) \ln(ax+b) - \frac{1}{3}\left(\frac{x^3}{3} - \frac{bx^2}{2a} + \frac{b^2 x}{a^2}\right).$

**6.** $\int x^3 \ln(ax+b)\, dx = \frac{1}{4}\left(x^4 - \frac{b^4}{a^4}\right) \ln(ax+b) -$

$$- \frac{1}{4}\left(\frac{x^4}{4} - \frac{bx^3}{3a} + \frac{b^2x^2}{2a^2} - \frac{b^3 x}{a^3}\right).$$

**7.** $\int \frac{\ln(ax+b)}{x^p}\, dx = -\frac{\ln(ax+b)}{(p-1)x^{p-1}} + \frac{a}{p-1} \int \frac{dx}{x^{p-1}(ax+b)}.$

**8.** $$\int \frac{\ln(ax+b)}{x^m}\,dx = -\frac{\ln(ax+b)}{(m-1)x^{m-1}} + \frac{1}{m-1}\left(-\frac{a}{b}\right)^{m-1}\left[\ln\frac{ax+b}{x} + \sum_{k=1}^{m-2}\frac{1}{k}\left(-\frac{b}{ax}\right)^k\right].$$

**9.** $$\int \frac{\ln(ax+b)}{x}\,dx = \ln|b|\cdot\ln|x| - \mathrm{Li}_2\left(-\frac{ax}{b}\right) \qquad [ab \neq 0].$$

**10.** $$= \ln|x|\cdot\ln(ax+b) - a\int\frac{\ln|x|\,dx}{ax+b}.$$

**11.** $$= \ln|b|\cdot|\ln x| + \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k^2}\left(\frac{ax}{b}\right)^k \qquad [|ax| < |b|].$$

**12.** $$= \frac{\ln^2|ax|}{2} + \sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\left(\frac{b}{ax}\right)^k \qquad [|ax| > |b|].$$

**13.** $$\int \frac{\ln(ax+b)}{x^2}\,dx = \frac{a}{b}\ln|x| - \left(\frac{1}{x}+\frac{a}{b}\right)\ln(ax+b).$$

## 6.5. Интегралы вида $\int x^p \ln\frac{x+a}{x-a}\,dx$.

**1.** $$\int x^p\ln\frac{x+a}{x-a}\,dx = \frac{x^{p+1}}{p+1}\ln\frac{x+a}{x-a} + \frac{2a}{p+1}\int\frac{x^{p+1}}{x^2-a^2}\,dx.$$

**2.** $$\int x^m\ln\frac{x+a}{x-a}\,dx = \frac{1}{m+1}\left[x^{m+1}-(-a)^{m+1}\right]\ln(x+a) - \frac{1}{m+1}\left(x^{m+1}-a^{m+1}\right)\ln(x-a) + \frac{a^{m+1}}{m+1}\sum_{k=1}^{m+1}\frac{1-(-1)^{m+k+1}}{k}\left(\frac{x}{a}\right)^k.$$

**3.** $$\int \ln\frac{x+a}{x-a}\,dx = (x+a)\ln(x+a) - (x-a)\ln(x-a) = x\ln\frac{x+a}{x-a} + a\ln\left(x^2-a^2\right).$$

**4.** $$\int x\ln\frac{x+a}{x-a}\,dx = \frac{x^2-a^2}{2}\ln\frac{x+a}{x-a} + ax.$$

**5.** $$\int x^2\ln\frac{x+a}{x-a}\,dx = \frac{x^3}{3}\ln\frac{x+a}{x-a} + \frac{a^3}{3}\ln\left(x^2-a^2\right) + \frac{ax^2}{3}.$$

**6.** $$\int x^3\ln\frac{x+a}{x-a}\,dx = \frac{x^4-a^4}{4}\ln\frac{x+a}{x-a} + \frac{ax^3}{6} + \frac{a^3x}{2}.$$

**7.** $$\int \frac{1}{x^m}\ln\frac{x+a}{x-a}\,dx = \frac{1}{(m-1)x^{m-1}}\ln\frac{x-a}{x+a} - \frac{1}{(m-1)a^{m-1}}\left[(-1)^m\ln\frac{x+a}{x} + \ln\frac{x-a}{x} + \sum_{k=1}^{m-2}\frac{1-(-1)^{m-k-1}}{k}\left(\frac{a}{x}\right)^k\right].$$

**8.** $$\int \frac{1}{x}\ln\frac{x+a}{x-a}\,dx = \mathrm{Li}_2\left(\frac{x}{a}\right) - \mathrm{Li}_2\left(-\frac{x}{a}\right) \qquad [a \neq 0].$$

**9.** $$= 2\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2}\left(\frac{x}{a}\right)^{2k+1}.$$

**10.** $\displaystyle\int \frac{1}{x^2}\ln\frac{x+a}{x-a}\,dx = \frac{1}{x}\ln\frac{x-a}{x+a} - \frac{1}{a}\ln\frac{x^2-a^2}{x^2}.$

## 6.6. Интегралы вида $\int x^{\pm m}\ln(x^2+a^2)\,dx$.

**1.** $$\int x^{2n}\ln(x^2+a^2)\,dx = \frac{1}{2n+1}\Big[x^{2n+1}\ln(x^2+a^2) - {} - 2\sum_{k=0}^{n}\frac{(-1)^{n-k}}{2k+1}\,a^{2n-2k}x^{2k+1} + (-1)^n 2a^{2n+1}\operatorname{arctg}\frac{x}{a}\Big].$$

**2.** $$\int x^{2n+1}\ln(x^2+a^2)\,dx = \frac{1}{2n+1}\Big\{[x^{2n+2}+(-1)^n a^{2n+2}]\ln(x^2+a^2) + \sum_{k=1}^{n+1}\frac{(-1)^{n-k}}{k}\,a^{2n-2k+2}x^{2k}\Big\}.$$

**3.** $$\int \ln(x^2+a^2)\,dx = x\ln(x^2+a^2) - 2x + 2a\operatorname{arctg}\frac{x}{a}.$$

**4.** $$\int x\ln(x^2+a^2)\,dx = \frac{1}{2}\,[(x^2+a^2)\ln(x^2+a^2) - x^2].$$

**5.** $$\int x^2\ln(x^2+a^2)\,dx = \frac{x^3}{3}\,\ln(x^2+a^2) - \frac{2}{9}\,x^3 + \frac{2}{3}\,a^2x - \frac{2}{3}a^3\operatorname{arctg}\frac{x}{a}.$$

**6.** $$\int x^3\ln(x^2+a^2)\,dx = \frac{1}{4}\Big[(x^4-a^4)\ln(x^2+a^2) + a^2x^2 - \frac{x^4}{2}\Big].$$

**7.** $$\int \frac{\ln(x^2+a^2)}{x^{2m}}\,dx = -\frac{\ln(x^2+a^2)}{(2m-1)x^{2m-1}} - {} - \frac{(-1)^m 2}{(2m-1)a^{2m-1}}\sum_{k=0}^{m-2}\frac{(-1)^k}{2k+1}\left(\frac{a}{x}\right)^{2k+1} - \frac{2(-1)^m}{(2m-1)a^{2m-1}}\operatorname{arctg}\frac{x}{a}.$$

**8.** $$\int \frac{\ln(x^2+a^2)}{x^{2m+1}}\,dx = -\frac{\ln(x^2+a^2)}{2mx^{2m}} + {} + \frac{(-1)^m}{2ma^{2m}}\Big[\ln\frac{x^2+a^2}{x^2} + \sum_{k=1}^{m-1}\frac{(-1)^k}{k}\left(\frac{a}{x}\right)^{2k}\Big].$$

**9.** $$\int \frac{\ln(x^2+a^2)}{x}\,dx = \ln^2|x| + \frac{1}{2}\,\mathrm{Li}_2\left(-\frac{a^2}{x^2}\right) \qquad [a\ne 0].$$

**10.** $$= \ln a^2\cdot\ln|x| - \frac{1}{2}\,\mathrm{Li}_2\left(-\frac{x^2}{a^2}\right) \qquad [a\ne 0].$$

**11.** $$\int \frac{\ln(x^2+a^2)}{x^2}\,dx = -\frac{\ln(x^2+a^2)}{x} + \frac{2}{a}\operatorname{arctg}\frac{x}{a}.$$

**12.** $$\int \frac{\ln(x^2+a^2)}{x^3}\,dx = -\frac{x^2+a^2}{2a^2x^2}\ln(x^2+a^2) + \frac{1}{a^2}\ln|x|.$$

**6.7.** Интегралы вида $\int x^{\pm m} \ln|x^2-a^2|\,dx$.

**1.** $$\int x^{2n} \ln|x^2-a^2|\,dx = \frac{1}{2n+1}\left[x^{2n+1}\ln|x^2-a^2| - {} \right.$$
$$\left. - 2\sum_{k=0}^{n} \frac{a^{2n-2k}}{2k+1}\,x^{2k+1} + a^{2n+1}\ln\left|\frac{x+a}{x-a}\right|\right].$$

**2.** $$\int x^{2n+1}\ln|x^2-a^2|\,dx =$$
$$= \frac{1}{2n+2}\left[(x^{2n+2}-a^{2n+2})\ln|x^2-a^2| - \sum_{k=1}^{n+1}\frac{a^{2n-2k+2}}{k}\,x^{2k}\right].$$

**3.** $$\int \ln|x^2-a^2|\,dx = x\ln|x^2-a^2| - 2x + a\ln\left|\frac{x+a}{x-a}\right|.$$

**4.** $$\int x\ln|x^2-a^2|\,dx = \frac{1}{2}\,[(x^2-a^2)\ln|x^2-a^2| - x^2].$$

**5.** $$\int x^2\ln|x^2-a^2|\,dx = \frac{x^3}{3}\,\ln|x^2-a^2| - \frac{2}{9}\,x^3 - \frac{2}{3}\,a^2x + \frac{a^3}{3}\ln\left|\frac{x+a}{x-a}\right|.$$

**6.** $$\int x^3\ln|x^2-a^2|\,dx = \frac{1}{4}\left[(x^4-a^4)\ln|x^2-a^2| - \frac{x^4}{2} - a^2x^2\right].$$

**7.** $$\int \frac{\ln|x^2-a^2|}{x^{2m}}\,dx = -\frac{\ln|x^2-a^2|}{(2m-1)x^{2m-1}} + {}$$
$$+ \frac{1}{(2m-1)a^{2m-1}}\sum_{k=0}^{m-2}\frac{1}{2k+1}\left(\frac{a}{x}\right)^{2k+1} - \frac{1}{(2m-1)a^{2m-1}}\ln\left|\frac{x+a}{x-a}\right|.$$

**8.** $$\int \frac{\ln|x^2-a^2|}{x^{2m+1}}\,dx = -\frac{\ln|x^2-a^2|}{2mx^{2m}} + \frac{1}{2ma^{2m}}\left[\ln\frac{|x^2-a^2|}{x^2} + \sum_{k=1}^{m-1}\frac{1}{k}\left(\frac{a}{x}\right)^{2k}\right].$$

**9.** $$\int \frac{\ln|x^2-a^2|}{x}\,dx = \ln^2|x| + \frac{1}{2}\,\mathrm{Li}_2\left(\frac{a^2}{x^2}\right) \qquad [x^2>a^2].$$

**10.** $$= \ln a^2\cdot\ln|x| - \frac{1}{2}\,\mathrm{Li}_2\left(\frac{x^2}{a^2}\right) \qquad [x^2<a^2].$$

**11.** $$\int \frac{\ln|x^2-a^2|}{x^2}\,dx = -\frac{\ln|x^2-a^2|}{x} - \frac{1}{a}\ln\left|\frac{x+a}{x-a}\right|.$$

**12.** $$\int \frac{\ln|x^2-a^2|}{x^3}\,dx = -\frac{x^2-a^2}{2a^2x^2}\ln|x^2-a^2| - \frac{1}{a^2}\ln|x|.$$

**6.8.** Интегралы вида $\int x^p \ln(x+\sqrt{x^2+a^2}\,)\,dx$.

**1.** $$\int x^p\ln(x+\sqrt{x^2+a^2}\,)\,dx = \frac{x^{p+1}}{p+1}\ln(x+\sqrt{x^2+a^2}\,) - \frac{1}{p+1}\int\frac{x^{p+1}\,dx}{\sqrt{x^2+a^2}}.$$

**2.** $$\int \ln(x+\sqrt{x^2+a^2}\,)\,dx = x\ln(x+\sqrt{x^2+a^2}\,) - \sqrt{x^2+a^2}\,.$$

**3.** $$\int x\ln(x+\sqrt{x^2+a^2}\,)\,dx = \left(\frac{x^2}{2}+\frac{a^2}{4}\right)\ln(x+\sqrt{x^2+a^2}\,) - \frac{x}{4}\sqrt{x^2+a^2}\,.$$

**4.** $\int x^2 \ln(x+\sqrt{x^2+a^2})\,dx =$

$$= \frac{x^3}{3}\ln(x+\sqrt{x^2+a^2}) - \frac{(x^2+a^2)^{3/2}}{9} + \frac{a^2}{3}\sqrt{x^2+a^2}.$$

**5.** $\int x^3 \ln(x+\sqrt{x^2+a^2})\,dx = \left(\frac{x^4}{4} - \frac{3a^4}{32}\right)\ln(x+\sqrt{x^2+a^2}) +$

$$+ \left(\frac{3}{32}a^2x - \frac{x^3}{16}\right)\sqrt{x^2+a^2}.$$

**6.** $\int \frac{\ln(x+\sqrt{x^2+a^2})}{x^{2n}}\,dx = -\frac{\ln(x+\sqrt{x^2+a^2})}{(2n-1)x^{2n-1}} - \frac{\sqrt{x^2+a^2}}{(2n-1)(2n-2)} \times$

$$\times \left[\frac{1}{a^2x^{2n-2}} + \frac{1}{a^{2n}}\sum_{k=1}^{n-2}(-1)^k \frac{(2n-3)(2n-5)\ldots(2n-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)}\left(\frac{a}{x}\right)^{2n-2k}\right] +$$

$$+ \frac{(-1)^n}{(2n-1)a^{2n-1}}\frac{(2n-3)!!}{(2n-2)!!}\ln\left|\frac{a+\sqrt{x^2+a^2}}{x}\right|.$$

**7.** $\int \frac{\ln(x+\sqrt{x^2+a^2})}{x^{2n+1}}\,dx =$

$$= -\frac{\ln(x+\sqrt{x^2+a^2})}{2nx^{2n}} - \frac{\sqrt{x^2+a^2}}{(2n-1)2n}\left[\frac{1}{a^2x^{2n-1}} +\right.$$

$$\left. + \frac{1}{a^{2n+1}}\sum_{k=1}^{n-1}(-1)^k \frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\left(\frac{a}{x}\right)^{2n-2k-1}\right].$$

**8.** $\int \frac{\ln(x+\sqrt{x^2+a^2})}{x}\,dx = \ln|a|\cdot\ln|x| + \frac{1}{2}\ln^2\frac{2x}{a} +$

$$+ \sum_{k=1}^{\infty}(-1)^k \frac{(2k-1)!!}{(2k-2)!!(2k)^3}\left(\frac{a}{x}\right)^{2k} \qquad [x/a > 1].$$

**9.** $$= \ln|a|\cdot\ln|x| - \frac{1}{2}\ln^2\left|\frac{2x}{a}\right| - \sum_{k=1}^{\infty}(-1)^k \frac{(2k-1)!!}{(2k-2)!!\,(2k)^3}\left(\frac{a}{x}\right)^{2k}$$

$$[x/a < -1].$$

**10.** $$= \ln|a|\cdot\ln|x| + \sum_{k=0}^{\infty}(-1)^k \frac{(2k+1)!!}{(2k)!!(2k+1)^3}\left(\frac{x}{a}\right)^{2k+1} \qquad [|x| < a].$$

**11.** $\int \frac{\ln(x+\sqrt{x^2+a^2})}{x^2}\,dx = -\frac{\ln(x+\sqrt{x^2+a^2})}{x} - \frac{1}{a}\ln\frac{a+\sqrt{x^2+a^2}}{|x|}.$

**12.** $\int \frac{\ln(x+\sqrt{x^2+a^2})}{x^3}\,dx = -\frac{\ln(x+\sqrt{x^2+a^2})}{2x^2} - \frac{\sqrt{x^2+a^2}}{2a^2x}.$

### 6.9. Интегралы вида $\int x^p \ln(x+\sqrt{x^2-a^2})\,dx$.

**1.** $\int x^p \ln(x+\sqrt{x^2-a^2})\,dx = \frac{x^{p+1}}{p+1}\ln(x+\sqrt{x^2-a^2}) - \frac{1}{p+1}\int \frac{x^{p+1}\,dx}{\sqrt{x^2-a^2}}.$

**2.** $\int \ln(x+\sqrt{x^2-a^2})\,dx = x\ln(x+\sqrt{x^2-a^2}) - \sqrt{x^2-a^2}.$

**3.** $\displaystyle\int x\ln\left(x+\sqrt{x^2-a^2}\,\right)dx=\left(\frac{x^2}{2}-\frac{a^2}{4}\right)\ln\left(x+\sqrt{x^2-a^2}\,\right)-\frac{x}{4}\sqrt{x^2-a^2}\,.$

**4.** $$\int x^2\ln\left(x+\sqrt{x^2-a^2}\,\right)dx=\frac{x^3}{3}\ln\left(x+\sqrt{x^2-a^2}\,\right)-\frac{(x^2-a^2)^{3/2}}{9}-\frac{a^2}{3}\sqrt{x^2-a^2}\,.$$

**5.** $$\int x^3\ln\left(x+\sqrt{x^2-a^2}\,\right)dx=\left(\frac{x^4}{4}-\frac{3a^4}{32}\right)\ln\left(x+\sqrt{x^2-a^2}\,\right)-\left(\frac{x^3}{16}+\frac{3}{32}a^2x\right)\sqrt{x^2-a^2}\,.$$

**6.** $$\int\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{x^{2n}}\,dx=-\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{(2n-1)x^{2n-1}}+\frac{\sqrt{x^2-a^2}}{(2n-1)(2n-2)}\times\left[\frac{1}{a^2x^{2n-2}}-\frac{1}{a^{2n}}\sum_{k=1}^{n-2}\frac{(2n-3)(2n-5)\ldots(2n-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)}\left(\frac{a}{x}\right)^{2n-2k}\right]+\frac{1}{(2n-1)a^{2n-1}}\,\frac{(2n-3)!!}{(2n-2)!!}\operatorname{arctg}\frac{\sqrt{x^2-a^2}}{a}.$$

**7.** $$\int\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{x^{2n+1}}\,dx=-\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{2nx^{2n}}+\frac{\sqrt{x^2-a^2}}{(2n-1)2n}\left[\frac{1}{a^2x^{2n-1}}+\frac{1}{a^{2n+1}}\sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\left(\frac{a}{x}\right)^{2n-2k-1}\right].$$

**8.** $$\int\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{x}\,dx=\ln|a|\cdot\ln|x|+\frac{1}{2}\ln^2\left|\frac{2x}{a}\right|+\sum_{k=1}^{\infty}\frac{(2k-1)!!}{(2k-2)!!(2k)^3}\left(\frac{a}{x}\right)^{2k}\qquad[|x|>|a|].$$

**9.** $\displaystyle\int\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{x^2}\,dx=-\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{x}+\frac{1}{a}\operatorname{arctg}\frac{\sqrt{x^2-a^2}}{a}.$

**10.** $\displaystyle\int\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{x^3}\,dx=-\frac{\ln\left(x+\sqrt{x^2-a^2}\,\right)}{2x^2}+\frac{\sqrt{x^2-a^2}}{2a^2x}.$

### 6.10. Интегралы вида $\displaystyle\int\frac{x^p}{\sqrt{x^2\pm a^2}}\ln\left(x+\sqrt{x^2\pm a^2}\,\right)dx$.

**1.** $$\int\frac{x^p}{\sqrt{x^2\pm a^2}}\ln\left(x+\sqrt{x^2\pm a^2}\,\right)dx=\frac{x^{p-1}}{p}\sqrt{x^2\pm a^2}\,\ln\left(x+\sqrt{x^2\pm a^2}\,\right)-\frac{x^p}{p^2}\mp\frac{(p-1)a^2}{p}\int\frac{x^{p-2}}{\sqrt{x^2\pm a^2}}\ln\left(x+\sqrt{x^2\pm a^2}\,\right)dx.$$

**2.** $\displaystyle\int\frac{1}{\sqrt{x^2\pm a^2}}\ln\left(x+\sqrt{x^2\pm a^2}\,\right)dx=\frac{1}{2}\ln^2\left(x+\sqrt{x^2\pm a^2}\,\right).$

**3.** $\displaystyle\int\frac{x}{\sqrt{x^2\pm a^2}}\ln\left(x+\sqrt{x^2\pm a^2}\,\right)dx=\sqrt{x^2\pm a^2}\,\ln\left(x+\sqrt{x^2\pm a^2}\,\right)-x.$

**6.11. Интегралы вида** $\int \ln^p(x+\sqrt{x^2\pm a^2}\,)\,dx$.

**1.** $$\int \ln^p(x+\sqrt{x^2\pm a^2}\,)\,dx = x\ln^p(x+\sqrt{x^2\pm a^2}\,) - \\ - p\sqrt{x^2\pm a^2}\,\ln^{p-1}(x+\sqrt{x^2\pm a^2}\,) + p(p-1)\int \ln^{p-2}(x+\sqrt{x^2\pm a^2}\,)\,dx.$$

**2.** $$\int \ln^n(x+\sqrt{x^2\pm a^2}\,)\,dx = \\ = \frac{x}{n+1}\sum_{k=0}^{[n/2]}(n+1)n\ldots(n-2k+1)\ln^{n-2k}(x+\sqrt{x^2\pm a^2}\,) - \\ - \sqrt{x^2\pm a^2}\sum_{k=0}^{[n/2]} n(n-1)\ldots(n-2k)\ln^{n-2k-1}(x+\sqrt{x^2\pm a^2}\,).$$

**3.** $$\int \ln^2(x+\sqrt{x^2\pm a^2}\,)\,dx = \\ = x\ln^2(x+\sqrt{x^2\pm a^2}\,) - 2\sqrt{x^2\pm a^2}\,\ln(x+\sqrt{x^2\pm a^2}\,) + 2x.$$

## 7. ОБРАТНЫЕ ТРИГОНОМЕТРИЧЕСКИЕ ФУНКЦИИ

### 7.1. Интегралы вида $\int \arcsin^p \frac{x}{a}\, dx$.

**1.** $$\int \arcsin^p \frac{x}{a}\, dx = x \arcsin^p \frac{x}{a} + p\sqrt{a^2 - x^2}\, \arcsin^{p-1} \frac{x}{a} - p(p-1) \int \arcsin^{p-2} \frac{x}{a}\, dx.$$

**2.** $$\int \arcsin^n \frac{x}{a}\, dx = x \sum_{k=0}^{[n/2]} (-1)^k (2k)!\, C_n^{2k} \arcsin^{n-2k} \frac{x}{a} - \sqrt{a^2 - x^2} \sum_{k=1}^{[(n+1)/2]} (-1)^k (2k-1)!\, C_n^{2k-1} \arcsin^{n-2k+1} \frac{x}{a}.$$

**3.** $$\int \arcsin \frac{x}{a}\, dx = x \arcsin \frac{x}{a} + \sqrt{a^2 - x^2}\,.$$

**4.** $$\int \arcsin^2 \frac{x}{a}\, dx = x \arcsin^2 \frac{x}{a} + 2\sqrt{a^2 - x^2}\, \arcsin \frac{x}{a} - 2x.$$

**5.** $$\int \arcsin^3 \frac{x}{a}\, dx = x \arcsin^3 \frac{x}{a} + 3\sqrt{a^2 - x^2}\, \arcsin^2 \frac{x}{a} - 6x \arcsin \frac{x}{a} - 6\sqrt{a^2 - x^2}\,.$$

**6.** $$\int \frac{dx}{\arcsin x} = \ln|\arcsin x| + \sum_{k=1}^{\infty} \frac{(-1)^k}{(2k)!\, 2k} \arcsin^{2k} x.$$

### 7.2. Интегралы вида $\int x^p \arcsin \frac{x}{a}\, dx$.

**1.** $$\int x^p \arcsin \frac{x}{a}\, dx = \frac{x^{p+1}}{p+1} \arcsin \frac{x}{a} - \frac{1}{p+1} \int \frac{x^{p+1}\, dx}{\sqrt{a^2 - x^2}}.$$

**2.** $$\int x^{2n} \arcsin \frac{x}{a}\, dx = \frac{x^{2n+1}}{2n+1} \arcsin \frac{x}{a} + \frac{\sqrt{a^2 - x^2}}{(2n+1)^2} \left[ x^{2n} + a^{2n} \sum_{k=0}^{n-1} \frac{2^{k+1} n(n-1)\ldots(n-k)}{(2n-1)(2n-3)\ldots(2n-2k-1)} \left(\frac{x}{a}\right)^{2n-2k-2} \right].$$

**3.** $$\int x^{2n+1} \arcsin \frac{x}{a}\, dx = \frac{x^{2n+2}}{2n+2} \arcsin \frac{x}{a} + \frac{\sqrt{a^2 - x^2}}{(2n+2)^2} \times \left[ x^{2n+1} + a^{2n+1} \sum_{k=1}^{n} \frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2^k n(n-1)\ldots(n-k+1)} \left(\frac{x}{a}\right)^{2n-2k+1} \right] - \frac{(2n+1)!!\, a^{2n+2}}{n!\, 2^{n+2} (n+1)^2} \arcsin \frac{x}{a}.$$

**4.** $$\int x \arcsin\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arcsin\frac{x}{a} + \frac{x}{4}\sqrt{a^2 - x^2}\,.$$

**5.** $$\int x^2 \arcsin\frac{x}{a}\,dx = \frac{x^3}{3}\arcsin\frac{x}{a} + \frac{x^2 + 2a^2}{9}\sqrt{a^2 - x^2}\,.$$

**6.** $$\int x^3 \arcsin\frac{x}{a}\,dx = \left(\frac{x^4}{4} - \frac{3a^4}{32}\right)\arcsin\frac{x}{a} + \frac{2x^3 + 3a^2 x}{32}\sqrt{a^2 - x^2}\,.$$

### 7.3. Интегралы вида $\int \frac{1}{x^p}\arcsin\frac{x}{a}\,dx$.

**1.** $$\int \frac{1}{x^p}\arcsin\frac{x}{a}\,dx = -\frac{1}{(p-1)x^{p-1}}\arcsin\frac{x}{a} + \frac{1}{p-1}\int\frac{dx}{x^{p-1}\sqrt{a^2 - x^2}}.$$

**2.** $$\int \frac{1}{x^{2n}}\arcsin\frac{x}{a}\,dx = -\frac{1}{(2n-1)x^{2n-1}}\arcsin\frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{(2n-1)(2n-2)}\times$$
$$\times\left[\frac{1}{a^2 x^{2n-2}} + \frac{1}{a^{2n}}\sum_{k=1}^{n-2}\frac{(2n-3)(2n-5)\ldots(2n-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)}\left(\frac{a}{x}\right)^{2n-2k-2}\right] -$$
$$-\frac{(2n-3)!!}{(n-1)!\,2^{n-1}(2n-1)a^{2n-1}}\ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right| \qquad [n \geqslant 2].$$

**3.** $$\int \frac{1}{x^{2n+1}}\arcsin\frac{x}{a}\,dx = -\frac{1}{2nx^{2n}}\arcsin\frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{2n(2n-1)}\times$$
$$\times\left[\frac{1}{a^2 x^{2n-1}} + \frac{1}{a^{2n+1}}\sum_{k=1}^{n-1}\frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)}\left(\frac{a}{x}\right)^{2n-2k-1}\right].$$

**4.** $$= -\frac{1}{2nx^{2n}}\arcsin\frac{x}{a} - \frac{1}{2na^{2n}}\sum_{k=0}^{n-1}\frac{C_{n-1}^k}{2n-2k-1}\left(\frac{a^2 - x^2}{x^2}\right)^{n-k-1/2}.$$

**5.** $$\int \frac{1}{x}\arcsin\frac{x}{a}\,dx = \sum_{k=0}^{\infty}\frac{(2k-1)!!}{(2k)!!(2k+1)^2}\left(\frac{x}{a}\right)^{2k+1}.$$

**6.** $$\int \frac{1}{x^2}\arcsin\frac{x}{a}\,dx = -\frac{1}{x}\arcsin\frac{x}{a} - \frac{1}{a}\ln\left|\frac{a + \sqrt{a^2 - x^2}}{x}\right|.$$

**7.** $$\int \frac{1}{x^3}\arcsin\frac{x}{a}\,dx = -\frac{1}{2x^2}\arcsin\frac{x}{a} - \frac{\sqrt{a^2 - x^2}}{2a^2 x}.$$

### 7.4. Интегралы вида $\int (a \pm x)^{\pm n + 1/2}\arcsin\frac{x}{a}\,dx$.

**1.** $$\int (a \pm x)^{n+1/2}\arcsin\frac{x}{a}\,dx = \pm\frac{2}{2n+3}(a \pm x)^{n+3/2}\arcsin\frac{x}{a} \pm$$
$$\pm\frac{2^{n+3}\sqrt{a \mp x}}{2n+3}\sum_{k=0}^{n+1}\frac{(-1)^k C_{n+1}^k}{2^k(2k+1)}a^{n-k+1}(a \mp x)^k.$$

**2.** $$\int \frac{1}{(a \pm x)^{n+3/2}}\arcsin\frac{x}{a}\,dx = \mp\frac{2}{(2n-1)(a \pm x)^{n+1/2}}\arcsin\frac{x}{a} -$$
$$-\frac{\sqrt{a \mp x}}{n(2n+1)a(a+x)^n}\left[1 + \sum_{k=1}^{n-1}\frac{(2n-1)(2n-3)\ldots(2n-2k+1)}{(2n-2)(2n-4)\ldots(2n-2k)}\left(\frac{a \pm x}{2a}\right)^k\right] +$$

$$+\frac{(2n-1)!!\,2^{3/2-n}}{(2n)!!\,(2n+1)a^{n+1/2}}\ln\frac{\sqrt{2a}-\sqrt{a\mp x}}{\sqrt{a\pm x}}.$$

## 7.5. Интегралы вида $\int x^p(a^2-x^2)^q\arcsin^r\frac{x}{a}\,dx$.

**1.** $$\int x^p(a^2-x^2)^{n+1/2}\arcsin\frac{x}{a}\,dx=-\frac{x^{p-1}(a^2-x^2)^{n+3/2}}{p+2n+2}\arcsin\frac{x}{a}+$$
$$+\frac{1}{p+2n+2}\sum_{k=0}^{n+1}\frac{(-1)^kC_{n+1}^k}{p+2k}a^{2n-2k+2}x^{p+2k}+$$
$$+\frac{(p-1)a^2}{p+2n+2}\int x^{p-2}(a^2-x^2)^{n+1/2}\arcsin\frac{x}{a}\,dx.$$

**2.** $$\int\sqrt{a^2-x^2}\,\arcsin\frac{x}{a}\,dx=\frac{a^2}{4}\arcsin^2\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}\,\arcsin\frac{x}{a}+\frac{a^2-x^2}{4}.$$

**3.** $$\int x(a^2-x^2)^{n+1/2}\arcsin\frac{x}{a}\,dx=$$
$$=-\frac{(a^2-x^2)^{n+3/2}}{2n+3}\arcsin\frac{x}{a}+\frac{1}{2n+3}\sum_{k=0}^{n+1}\frac{(-1)^kC_{n+1}^k}{2k+1}a^{2n-2k+2}x^{2k+1}.$$

**4.** $$\int(a^2-x^2)^p\arcsin\frac{x}{a}\,dx=$$
$$=\frac{x(a^2-x^2)^p}{2p+1}\arcsin\frac{x}{a}+\frac{(a^2-x^2)^{p+1/2}}{(2p+1)^2}+\frac{2a^2p}{2p+1}\int(a^2-x^2)^{p-1}\arcsin\frac{x}{a}\,dx.$$

**5.** $$\int\frac{1}{\sqrt{a^2-x^2}}\arcsin^p\frac{x}{a}\,dx=\frac{1}{p+1}\arcsin^{p+1}\frac{x}{a}.$$

**6.** $$\int\frac{dx}{\sqrt{a^2-x^2}\,\arcsin\frac{x}{a}}=\ln\left|\arcsin\frac{x}{a}\right|.$$

**7.** $$\int\frac{\sqrt{a^2-x^2}}{x}\arcsin\frac{x}{a}\,dx=\sqrt{a^2-x^2}\,\arcsin\frac{x}{a}-x+$$
$$+a^2\int\frac{1}{x\sqrt{a^2-x^2}}\arcsin\frac{x}{a}\,dx.$$

**8.** $$\int\frac{(a^2-x^2)^{n+1/2}}{x^p}\arcsin\frac{x}{a}\,dx=-\frac{(a^2-x^2)^{n+3/2}}{(p-1)a^2x^{p-1}}\arcsin\frac{x}{a}+\frac{1}{p-1}\times$$
$$\times\sum_{k=0}^{n+1}\frac{(-1)^kC_{n+1}^k}{2k-p+2}a^{2n-2k}x^{2k-p+2}-\frac{2n-p+4}{(p-1)a^2}\int\frac{(a^2-x^2)^{n+1/2}}{x^{p-2}}\arcsin\frac{x}{a}\,dx.$$

**9.** $$=-\frac{(a^2-x^2)^{n+1/2}}{(p-1)x^{p-1}}\arcsin\frac{x}{a}+$$
$$+\frac{1}{p-1}\sum_{k=0}^{n}\frac{(-1)^kC_n^k}{2k-p+2}a^{2n-2k}x^{2k-p+2}-\frac{2n+1}{p-1}\int\frac{(a^2-x^2)^{n-1/2}}{x^{p-2}}\arcsin\frac{x}{a}\,dx.$$

Если $k=p/2-1$, то соответствующий член в первой сумме следует заменить на $(-1)^{p/2-1}a^{2n-p+2}C_{n+1}^{p/2-1}\ln|x|$, а во второй сумме — на $(-1)^{p/2-1}a^{2n-p+2}\times$ $\times\,C_n^{p/2-1}\ln|x|$.

**10.** $$\int \frac{(a^2-x^2)^{n+1/2}}{x} \arcsin\frac{x}{a}\,dx = \frac{(a^2-x^2)^{n+1/2}}{2n+1}\arcsin\frac{x}{a} - \frac{1}{2n+1}\sum_{k=0}^{n}\frac{(-1)^k C_n^k}{2k+1}\,a^{2n-2k}x^{2k+1} + a^2\int\frac{(a^2-x^2)^{n-1/2}}{x}\arcsin\frac{x}{a}\,dx.$$

**11.** $$\int \frac{1}{(a^2-x^2)^p}\arcsin\frac{x}{a}\,dx = \frac{x}{2a^2(p-1)(a^2-x^2)^{p-1}}\arcsin\frac{x}{a} - \frac{1}{(2p-2)(2p-3)a^2(a^2-x^2)^{p-3/2}} + \frac{2p-3}{(2p-2)a^2}\int\frac{1}{(a^2-x^2)^{p-1}}\arcsin\frac{x}{a}\,dx.$$

**12.** $$\int \frac{1}{(a^2-x^2)^{3/2}}\arcsin\frac{x}{a}\,dx = \frac{x}{a^2\sqrt{a^2-x^2}}\arcsin\frac{x}{a} + \frac{1}{a^2}\ln\sqrt{a^2-x^2}\,.$$

**13.** $$\int \frac{x^p}{\sqrt{a^2-x^2}}\arcsin\frac{x}{a}\,dx = -\frac{x^{p-1}}{p}\sqrt{a^2-x^2}\arcsin\frac{x}{a} + \frac{x^p}{p^2} + \frac{(p-1)a^2}{p}\int\frac{x^{p-2}}{\sqrt{a^2-x^2}}\arcsin\frac{x}{a}\,dx.$$

**14.** $$\int \frac{x}{\sqrt{a^2-x^2}}\arcsin\frac{x}{a}\,dx = -\sqrt{a^2-x^2}\arcsin\frac{x}{a} + x.$$

**15.** $$\int \frac{x^2}{\sqrt{a^2-x^2}}\arcsin\frac{x}{a}\,dx = \frac{a^2}{4}\arcsin^2\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}\arcsin\frac{x}{a} + \frac{x^2}{4}.$$

**16.** $$\int \frac{x}{(a^2-x^2)^q}\arcsin\frac{x}{a}\,dx = \frac{1}{2(q-1)(a^2-x^2)^{q-1}}\arcsin\frac{x}{a} - \frac{1}{2(q-1)}\int\frac{dx}{(a^2-x^2)^{q-1/2}}.$$

**17.** $$\int \frac{x^p}{(a^2-x^2)^q}\arcsin\frac{x}{a}\,dx = \frac{x^{p-1}}{(2q-p-1)(a^2-x^2)^{q-1}}\arcsin\frac{x}{a} - \frac{1}{2q-p-1}\int\frac{x^{p-1}\,dx}{(a^2-x^2)^{q-1/2}} - \frac{(p-1)a^2}{2q-p-1}\int\frac{x^{p-2}}{(a^2-x^2)^q}\arcsin\frac{x}{a}\,dx.$$

**18.** $$= \frac{x^{p-1}}{2(q-1)(a^2-x^2)^{q-1}}\arcsin\frac{x}{a} - \frac{1}{2(q-1)}\int\frac{x^{p-1}\,dx}{(a^2-x^2)^{q-1/2}} - \frac{p-1}{2(q-1)}\int\frac{x^{p-2}}{(a^2-x^2)^{q-1}}\arcsin\frac{x}{a}\,dx.$$

**19.** $$\int \frac{1}{x^p(a^2-x^2)^q}\arcsin\frac{x}{a}\,dx = -\frac{1}{(p-1)a^2x^{p-1}(a^2-x^2)^{q-1}}\arcsin\frac{x}{a} + \frac{1}{(p-1)a^2}\int\frac{dx}{x^{p-1}(a^2-x^2)^{q-1/2}} + \frac{p+2q-3}{(p-1)a^2}\int\frac{1}{x^{p-2}(a^2-x^2)^q}\arcsin\frac{x}{a}\,dx.$$

**20.** $$\int \frac{1}{x(a^2-x^2)^q}\arcsin\frac{x}{a}\,dx = \frac{1}{2(q-1)a^2(a^2-x^2)^{q-1}}\arcsin\frac{x}{a} - \frac{1}{2(q-1)a^2}\int\frac{dx}{(a^2-x^2)^{q-1/2}} + \frac{1}{a^2}\int\frac{1}{x(a^2-x^2)^{q-1}}\arcsin\frac{x}{a}\,dx.$$

**7.6. Интегралы вида** $\displaystyle\int \arccos^p \frac{x}{a}\,dx$.

**1.** $$\int \arccos^p \frac{x}{a}\,dx = x\arccos^p\frac{x}{a} - p\sqrt{a^2-x^2}\,\arccos^{p-1}\frac{x}{a} - p(p-1)\int \arccos^{p-2}\frac{x}{a}\,dx.$$

**2.** $$\int \arccos^n \frac{x}{a}\,dx = x\sum_{k=0}^{[n/2]}(-1)^k(2k)!\,C_n^{2k}\arccos^{n-2k}\frac{x}{a} + \sqrt{a^2-x^2}\sum_{k=1}^{[(n+1)/2]}(-1)^k(2k-1)!\,C_n^{2k-1}\arccos^{n-2k+1}\frac{x}{a}.$$

**3.** $$\int \arccos\frac{x}{a}\,dx = x\arccos\frac{x}{a} - \sqrt{a^2-x^2}\,.$$

**4.** $$\int \arccos^2\frac{x}{a}\,dx = x\arccos^2\frac{x}{a} - 2\sqrt{a^2-x^2}\,\arccos\frac{x}{a} - 2x.$$

**5.** $$\int \arccos^3\frac{x}{a}\,dx = x\arccos^3\frac{x}{a} - 3\sqrt{a^2-x^2}\,\arccos^2\frac{x}{a} - 6x\arccos\frac{x}{a} + 6\sqrt{a^2-x^2}\,.$$

**6.** $$\int \frac{dx}{\arccos x} = \sum_{k=0}^{\infty}\frac{(-1)^{k+1}}{(2k+1)!\,(2k+1)}\arccos^{2k+1}x.$$

**7.7. Интегралы вида** $\displaystyle\int x^p\arccos\frac{x}{a}\,dx$.

**1.** $$\int x^p\arccos\frac{x}{a}\,dx = \frac{x^{p+1}}{p+1}\arccos\frac{x}{a} + \frac{1}{p+1}\int\frac{x^{p+1}\,dx}{\sqrt{a^2-x^2}}.$$

**2.** $$\int x^{2n}\arccos\frac{x}{a}\,dx = \frac{x^{2n+1}}{2n+1}\arccos\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{(2n+1)^2}\left[x^{2n} + a^{2n}\sum_{k=0}^{n-1}\frac{2^{k+1}n(n-1)\ldots(n-k)}{(2n-1)(2n-3)\ldots(2n-2k-1)}\left(\frac{x}{a}\right)^{2n-2k-2}\right].$$

**3.** $$\int x^{2n+1}\arccos\frac{x}{a}\,dx = \frac{x^{2n+2}}{2n+2}\arccos\frac{x}{a} - \frac{\sqrt{a^2-x^2}}{(2n+2)^2}\times\left[x^{2n+1} + a^{2n+1}\sum_{k=1}^{n}\frac{(2n+1)(2n-1)\ldots(2n-2k+3)}{2^k n(n-1)\ldots(n-k+1)}\left(\frac{x}{a}\right)^{2n-2k+1}\right] - \frac{(2n+1)!!\,a^{2n+2}}{n!\,2^{n+2}(n+1)^2}\arccos\frac{x}{a}.$$

**4.** $$\int x\arccos\frac{x}{a}\,dx = \left(\frac{x^2}{2} - \frac{a^2}{4}\right)\arccos\frac{x}{a} - \frac{x}{4}\sqrt{a^2-x^2}\,.$$

**5.** $$\int x^2\arccos\frac{x}{a}\,dx = \frac{x^3}{3}\arccos\frac{x}{a} - \frac{x^2+2a^2}{9}\sqrt{a^2-x^2}\,.$$

**6.** $$\int x^3\arccos\frac{x}{a}\,dx = \left(\frac{x^4}{4} - \frac{3a^4}{32}\right)\arccos\frac{x}{a} - \frac{2x^3+3a^2x}{32}\sqrt{a^2-x^2}\,.$$

### 7.8. Интегралы вида $\int \frac{1}{x^p} \arccos \frac{x}{a}\, dx$.

**1.** $$\int \frac{1}{x^p} \arccos \frac{x}{a}\, dx = -\frac{1}{(p-1)x^{p-1}} \arccos \frac{x}{a} - \frac{1}{p-1} \int \frac{dx}{x^{p-1}\sqrt{a^2-x^2}}.$$

**2.** $$\int \frac{1}{x^{2n}} \arccos \frac{x}{a}\, dx = -\frac{1}{(2n-1)x^{2n-1}} \arccos \frac{x}{a} + \frac{\sqrt{a^2-x^2}}{(2n-1)(2n-2)} \left[ \frac{1}{a^2 x^{2n-2}} + \frac{1}{a^{2n}} \sum_{k=1}^{n-2} \frac{(2n-3)(2n-5)\ldots(2n-2k-1)}{2^k(n-2)(n-3)\ldots(n-k-1)} \times \left(\frac{a}{x}\right)^{2n-2k-2} \right] + \frac{(2n-3)!!}{(n-1)!\, 2^{n-1}(2n-1)a^{2n-1}} \ln \left| \frac{a+\sqrt{a^2-x^2}}{x} \right| \quad [n \geqslant 2].$$

**3.** $$\int \frac{1}{x^{2n+1}} \arccos \frac{x}{a}\, dx = -\frac{1}{2nx^{2n}} \arccos \frac{x}{a} + \frac{\sqrt{a^2-x^2}}{2n(2n-1)} \left[ \frac{1}{a^2 x^{2n-1}} + \frac{1}{a^{2n+1}} \sum_{k=1}^{n-1} \frac{2^k(n-1)(n-2)\ldots(n-k)}{(2n-3)(2n-5)\ldots(2n-2k-1)} \left(\frac{a}{x}\right)^{2n-2k-1} \right].$$

**4.** $$= -\frac{1}{2nx^{2n}} \arccos \frac{x}{a} + \frac{1}{2na^{2n}} \sum_{k=0}^{n-1} \frac{C_{n-1}^k}{2n-2k-1} \left( \frac{a^2-x^2}{x^2} \right)^{n-k-1/2}.$$

**5.** $$\int \frac{1}{x} \arccos \frac{x}{a}\, dx = -\sum_{k=0}^{\infty} \frac{(2k-1)!!}{(2k)!!(2k+1)^2} \left(\frac{x}{a}\right)^{2k+1} + \frac{\pi}{2} \ln |x|.$$

**6.** $$\int \frac{1}{x^2} \arccos \frac{x}{a}\, dx = -\frac{1}{x} \arccos \frac{x}{a} + \frac{1}{a} \ln \left| \frac{a+\sqrt{a^2-x^2}}{x} \right|.$$

**7.** $$\int \frac{1}{x^3} \arccos \frac{x}{a}\, dx = -\frac{1}{2x^2} \arccos \frac{x}{a} + \frac{\sqrt{a^2-x^2}}{2a^2 x}.$$

### 7.9. Интегралы вида $\int (a \pm x)^{\pm n+1/2} \arccos \frac{x}{a}\, dx$.

**1.** $$\int (a \pm x)^{n+1/2} \arccos \frac{x}{a}\, dx = \pm \frac{2}{2n+3} (a \pm x)^{n+3/2} \arccos \frac{x}{a} - \frac{2^{n+3}\sqrt{a \mp x}}{2n+3} \sum_{k=0}^{n+1} \frac{(-1)^k C_{n+1}^k}{2^k(2k+1)} a^{n-k+1} (a \mp x)^k.$$

**2.** $$\int \frac{1}{(a \pm x)^{n+3/2}} \arccos \frac{x}{a}\, dx = \mp \frac{2}{(2n-1)(a \pm x)^{n+1/2}} \arccos \frac{x}{a} + \frac{\sqrt{a \mp x}}{n(2n+1)a(a \pm x)^n} \times \left[ 1 + \sum_{k=1}^{n-1} \frac{(2n-3)(2n-5)\ldots(2n-2k-1)}{(2n-4)(2n-6)\ldots(2n-2k-2)} \left( \frac{a \pm x}{2a} \right)^k \right] - \frac{(2n-1)!!\, 2^{3/2-n}}{(2n)!!\, (2n+1)a^{n+1/2}} \ln \frac{\sqrt{2a} - \sqrt{a \mp x}}{\sqrt{a \pm x}}.$$

**7.10. Интегралы вида** $\int x^p (a^2-x^2)^q \arccos^r \frac{x}{a}\, dx$.

**1.** $$\int x^p (a^2-x^2)^{n+1/2} \arccos\frac{x}{a}\, dx = -\frac{x^{p-1}(a^2-x^2)^{n+3/2}}{p+2n+2}\arccos\frac{x}{a} - \frac{1}{p+2n+2}\sum_{k=0}^{n+1}\frac{(-1)^k C_{n+1}^k}{p+2k}\, a^{2n-2k+2}x^{p+2k} + \frac{(p-1)a^2}{p+2n+2}\int x^{p-2}(a^2-x^2)^{n+1/2}\arccos\frac{x}{a}\, dx.$$

**2.** $$\int \sqrt{a^2-x^2}\,\arccos\frac{x}{a}\, dx = -\frac{a^2}{4}\arccos^2\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}\,\arccos\frac{x}{a} - \frac{a^2-x^2}{4}.$$

**3.** $$\int x(a^2-x^2)^{n+1/2}\arccos\frac{x}{a}\, dx = -\frac{(a^2-x^2)^{n+3/2}}{2n+3}\arccos\frac{x}{a} - \frac{1}{2n+3}\sum_{k=0}^{n+1}\frac{(-1)^k C_{n+1}^k}{2k+1}a^{2n-2k+2}x^{2k+1}.$$

**4.** $$\int (a^2-x^2)^p \arccos\frac{x}{a}\, dx = \frac{x(a^2-x^2)^p}{2p+1}\arccos\frac{x}{a} - \frac{(a^2-x^2)^{p+1/2}}{(2p+1)^2} + \frac{2a^2p}{2p+1}\int (a^2-x^2)^{p-1}\arccos\frac{x}{a}\, dx.$$

**5.** $$\int \frac{1}{\sqrt{a^2-x^2}}\arccos^p\frac{x}{a}\, dx = -\frac{1}{p+1}\arccos^{p+1}\frac{x}{a}.$$

**6.** $$\int \frac{dx}{\sqrt{a^2-x^2}\,\arccos\frac{x}{a}} = -\ln\left|\arccos\frac{x}{a}\right|.$$

**7.** $$\int \frac{\sqrt{a^2-x^2}}{x}\arccos\frac{x}{a}\, dx = \sqrt{a^2-x^2}\,\arccos\frac{x}{a} + x + \int \frac{a^2}{x\sqrt{a^2-x^2}}\arccos\frac{x}{a}\, dx.$$

**8.** $$\int \frac{(a^2-x^2)^{n+1/2}}{x^p}\arccos\frac{x}{a}\, dx = -\frac{(a^2-x^2)^{n+3/2}}{(p-1)a^2x^{p-1}}\arccos\frac{x}{a} - \frac{1}{p-1}\times \sum_{k=0}^{n+1}\frac{(-1)^k C_{n+1}^k}{2k-p+2}a^{2n-2k}x^{2k-p+2} - \frac{2n-p+4}{(p-1)a^2}\int \frac{(a^2-x^2)^{n+1/2}}{x^{p-2}}\arccos\frac{x}{a}\, dx.$$

**9.** $$= -\frac{(a^2-x^2)^{n+1/2}}{(p-1)x^{p-1}}\arccos\frac{x}{a} - \frac{1}{p-1}\sum_{k=0}^{n}\frac{(-1)^k C_n^k}{2k-p+2}a^{2n-2k}x^{2k-p+2} - \frac{2n+1}{p-1}\int \frac{(a^2-x^2)^{n-1/2}}{x^{p-2}}\arccos\frac{x}{a}\, dx.$$

Если $k=p/2-1$, то соответствующий член в первой сумме следует заменить на $(-1)^{p/2+1}a^{2n-p+2}C_{n+1}^{p/2-1}\ln|x|$, а во второй сумме — на $(-1)^{p/2+1}a^{2n-p+2}\times C_n^{p/2-1}\ln|x|$.

**10.** $$\int \frac{(a^2-x^2)^{n+1/2}}{x} \arccos\frac{x}{a}\,dx = \frac{(a^2-x^2)^{n+1/2}}{2n+1}\arccos\frac{x}{a} + \frac{1}{2n+1}\sum_{k=0}^{n}\frac{(-1)^k C_n^k}{2k+1}a^{2n-2k}x^{2k+1} + a^2\int\frac{(a^2-x^2)^{n-1/2}}{x}\arccos\frac{x}{a}\,dx.$$

**11.** $$\int \frac{1}{(a^2-x^2)^p}\arccos\frac{x}{a}\,dx = \frac{x}{2a^2(p-1)(a^2-x^2)^{p-1}}\arccos\frac{x}{a} + \frac{1}{(2p-2)(2p-3)a^2(a^2-x^2)^{p-3/2}} + \frac{2p-3}{(2p-2)a^2}\int\frac{1}{(a^2-x^2)^{p-1}}\arccos\frac{x}{a}\,dx.$$

**12.** $$\int \frac{1}{(a^2-x^2)^{3/2}}\arccos\frac{x}{a}\,dx = \frac{x}{a^2\sqrt{a^2-x^2}}\arccos\frac{x}{a} - \frac{1}{a^2}\ln\sqrt{a^2-x^2}\,.$$

**13.** $$\int \frac{x^p}{\sqrt{a^2-x^2}}\arccos\frac{x}{a}\,dx = -\frac{x^{p-1}}{p}\sqrt{a^2-x^2}\arccos\frac{x}{a} - \frac{x^p}{p^2} + \frac{(p-1)a^2}{p}\int\frac{x^{p-2}}{\sqrt{a^2-x^2}}\arccos\frac{x}{a}\,dx.$$

**14.** $$\int \frac{x}{\sqrt{a^2-x^2}}\arccos\frac{x}{a}\,dx = -\sqrt{a^2-x^2}\arccos\frac{x}{a} - x.$$

**15.** $$\int \frac{x^2}{\sqrt{a^2-x^2}}\arccos\frac{x}{a}\,dx = -\frac{a^2}{4}\arccos^2\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2}\arccos\frac{x}{a} - \frac{x^2}{4}.$$

**16.** $$\int \frac{x}{(a^2-x^2)^p}\arccos\frac{x}{a}\,dx = \frac{1}{2(p-1)(a^2-x^2)^{p-1}}\arccos\frac{x}{a} + \frac{1}{2(p-1)}\int\frac{dx}{(a^2-x^2)^{p-1/2}}.$$

**17.** $$\int \frac{x^p}{(a^2-x^2)^q}\arccos\frac{x}{a}\,dx = \frac{x^{p-1}}{(2q-p-1)(a^2-x^2)^{q-1}}\arccos\frac{x}{a} + \frac{1}{2q-p-1}\int\frac{x^{p-1}\,dx}{(a^2-x^2)^{q-1/2}} - \frac{(p-1)a^2}{2q-p-1}\int\frac{x^{p-2}}{(a^2-x^2)^q}\arccos\frac{x}{a}\,dx.$$

**18.** $$= \frac{x^{p-1}}{2(q-1)(a^2-x^2)^{q-1}}\arccos\frac{x}{a} + \frac{1}{2(q-1)}\int\frac{x^{p-1}\,dx}{(a^2-x^2)^{q-1/2}} - \frac{p-1}{2(q-1)}\int\frac{x^{p-2}}{(a^2-x^2)^{q-1}}\arccos\frac{x}{a}\,dx.$$

**19.** $$\int \frac{1}{x^p(a^2-x^2)^q}\arccos\frac{x}{a}\,dx = -\frac{1}{(p-1)a^2x^{p-1}(a^2-x^2)^{q-1}}\arccos\frac{x}{a} - \frac{1}{(p-1)a^2}\int\frac{dx}{x^{p-1}(a^2-x^2)^{q-1/2}} + \frac{p+2q-3}{(p-1)a^2}\int\frac{1}{x^{p-2}(a^2-x^2)^q}\arccos\frac{x}{a}\,dx.$$

**20.** $$\int \frac{1}{x(a^2-x^2)^q}\arccos\frac{x}{a}\,dx = \frac{1}{2(q-1)a^2(a^2-x^2)^{q-1}}\arccos\frac{x}{a} + \frac{1}{2(q-1)a^2}\int\frac{dx}{(a^2-x^2)^{q-1/2}} + \frac{1}{a^2}\int\frac{1}{x(a^2-x^2)^{q-1}}\arccos\frac{x}{a}\,dx.$$

### 7.11. Интегралы вида $\int x^p \operatorname{arctg}\frac{x}{a}\,dx$.

**1.** $$\int x^p \operatorname{arctg}\frac{x}{a}\,dx = \frac{x^{p+1}}{p+1}\operatorname{arctg}\frac{x}{a} - \frac{a}{p+1}\int\frac{x^{p+1}\,dx}{x^2+a^2}.$$

**2.** $$\int x^{2n}\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^{2n+1}}{2n+1}\operatorname{arctg}\frac{x}{a} - \frac{a}{2(2n+1)}\left[n!\sum_{k=0}^{n-1}\frac{(-a^2)^k(x^2+a^2)^{n-k}}{(n-k)!k!(n-k)} + (-a^2)^n\ln(x^2+a^2)\right].$$

**3.** $$\int x^{2n+1}\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^{2n+2}}{2n+2}\operatorname{arctg}\frac{x}{a} + \frac{a^{2n+2}}{2n+2}\left[\sum_{k=1}^{n+1}\frac{(-1)^{n+k}}{2k-1}\left(\frac{x}{a}\right)^{2k-1} + (-1)^n\operatorname{arctg}\frac{x}{a}\right].$$

**4.** $$\int \operatorname{arctg}\frac{x}{a}\,dx = x\operatorname{arctg}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2).$$

**5.** $$\int x\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^2+a^2}{2}\operatorname{arctg}\frac{x}{a} - \frac{ax}{2}.$$

**6.** $$\int x^2\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arctg}\frac{x}{a} - \frac{ax^2}{6} + \frac{a^3}{6}\ln(x^2+a^2).$$

**7.** $$\int x^3\operatorname{arctg}\frac{x}{a}\,dx = \frac{x^4-a^4}{4}\operatorname{arctg}\frac{x}{a} - \frac{ax^3}{12} + \frac{a^3x}{4}.$$

### 7.12. Интегралы вида $\int \frac{1}{x^p}\operatorname{arctg}\frac{x}{a}\,dx$.

**1.** $$\int \frac{1}{x^p}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{1}{(p-1)x^{p-1}}\operatorname{arctg}\frac{x}{a} + \frac{a}{p-1}\int\frac{dx}{x^{p-1}(x^2+a^2)}.$$

**2.** $$\int \frac{1}{x^{2n}}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{1}{(2n-1)x^{2n-1}}\operatorname{arctg}\frac{x}{a} - \frac{1}{2(2n-1)a^{2n-1}}\left[\sum_{k=0}^{n-2}\frac{(-1)^{n-k}}{k+1}\left(\frac{a}{x}\right)^{2k+2} + (-1)^{n-1}\ln\frac{x^2+a^2}{x^2}\right].$$

**3.** $$\int \frac{1}{x^{2n+1}}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{1}{2nx^{2n}}\operatorname{arctg}\frac{x}{a} + \frac{1}{2na^{2n}}\left[\sum_{k=1}^{n}\frac{(-1)^k}{2n-2k+1}\left(\frac{a}{x}\right)^{2n-2k+1} + (-1)^n\operatorname{arctg}\frac{x}{a}\right].$$

**4.** $$\int \frac{1}{x}\operatorname{arctg}\frac{x}{a}\,dx = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\left(\frac{x}{a}\right)^{2k+1} \qquad [|x|<|a|].$$

**5.** $$= \frac{\pi}{2}\ln|x| + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\left(\frac{a}{x}\right)^{2k+1} \qquad [x/a>1].$$

**6.** $$= -\frac{\pi}{2}\ln|x| + \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\left(\frac{a}{x}\right)^{2k+1} \qquad [x/a<-1].$$

**7.** $$\int \frac{1}{x^2}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{1}{x}\operatorname{arctg}\frac{x}{a} + \frac{1}{2a}\ln\frac{x^2}{x^2+a^2}.$$

**8.** $$\int \frac{1}{x^3}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{1}{2}\left(\frac{1}{x^2}+\frac{1}{a^2}\right)\operatorname{arctg}\frac{x}{a} - \frac{1}{2ax}.$$

**9.** $\displaystyle\int \frac{1}{(x+b)^2} \operatorname{arctg}\frac{x}{a}\,dx = \frac{a}{a^2+b^2}\left[\ln\frac{x+b}{\sqrt{x^2+a^2}} - \frac{a^2-bx}{a(x+b)}\operatorname{arctg}\frac{x}{a}\right].$

## 7.13. Интегралы вида $\displaystyle\int x^p(x^2+a^2)^q \operatorname{arctg}\frac{x}{a}\,dx$.

**1.** $\displaystyle\int x^p(x^2+a^2)^n \operatorname{arctg}\frac{x}{a}\,dx =$

$$= \operatorname{arctg}\frac{x}{a}\sum_{k=0}^{n} C_n^k \frac{a^{2n-2k}x^{p+2k+1}}{p+2k+1} - \sum_{k=0}^{n} C_n^k \frac{a^{2n-2k+1}}{p+2k+1}\int \frac{x^{p+2k+1}}{x^2+a^2}\,dx.$$

Если $k = -\dfrac{p+1}{2}$ $(k = 0, 1, \ldots, n)$, то соответствующие члены сумм заменяются на $C_n^k a^{p+2n+1}\ln\dfrac{x}{a}\operatorname{arctg}\dfrac{x}{a}$ и $C_n^k a^{p+2n+2}\displaystyle\int\frac{1}{x^2+a^2}\ln\frac{x}{a}\,dx$.

**2.** $\displaystyle\int \frac{x^p}{(x^2+a^2)^q}\operatorname{arctg}\frac{x}{a}\,dx = -\frac{x^{p-1}}{2(q-1)(x^2+a^2)^{q-1}}\operatorname{arctg}\frac{x}{a} +$

$$+ \frac{a}{2(q-1)}\int\frac{x^{p-1}\,dx}{(x^2+a^2)^q} + \frac{p-1}{2(q-1)}\int\frac{x^{p-2}}{(x^2+a^2)^{q-1}}\operatorname{arctg}\frac{x}{a}\,dx.$$

**3.** $\displaystyle\int \frac{1}{(x^2+a^2)^q}\operatorname{arctg}\frac{x}{a}\,dx = \frac{x}{2(q-1)a^2(x^2+a^2)^{q-1}}\operatorname{arctg}\frac{x}{a} +$

$$+ \frac{1}{4(q-1)^2a(x^2+a^2)^{q-1}} + \frac{2q-3}{(2q-2)a^2}\int\frac{1}{(x^2+a^2)^{q-1}}\operatorname{arctg}\frac{x}{a}\,dx.$$

**4.** $\displaystyle\int \frac{1}{(x^2+a^2)^n}\operatorname{arctg}\frac{x}{a}\,dx = \operatorname{arctg}\frac{x}{a}\,\times$

$$\times\left[\sum_{k=1}^{n-1}\frac{(2n-2k-2)!!(2n-3)!!}{(2n-2)!!(2n-2k-1)!!}\cdot\frac{x}{a^{2k}(x^2+a^2)^{n-k}} + \frac{1}{2a^{2n-1}}\frac{(2n-3)!!}{(2n-2)!!}\right] +$$

$$+ \frac{a}{2}\sum_{k=1}^{n-1}\frac{(2n-3)!!(2n-2k-2)!!}{(2n-2)!!(2n-2k-1)!!(n-k)}\,\frac{1}{a^{2k}(x^2+a^2)^{n-k}}.$$

**5.** $\displaystyle\int\frac{x^p}{x^2+a^2}\operatorname{arctg}\frac{x}{a}\,dx = \int x^{p-2}\operatorname{arctg}\frac{x}{a}\,dx - a^2\int\frac{x^{p-2}}{x^2+a^2}\operatorname{arctg}\frac{x}{a}\,dx.$

**6.** $\displaystyle\int\frac{x^{2m}}{x^2+a^2}\operatorname{arctg}\frac{x}{a}\,dx = \frac{(-1)^m}{2}\,a^{2m-1}\operatorname{arctg}^2\frac{x}{a} +$

$$+ \operatorname{arctg}\frac{x}{a}\sum_{k=1}^{m}\frac{(-1)^{m-k}}{2k-1}\,a^{2m-2k}x^{2k-1} - \sum_{k=1}^{m}\frac{(-1)^{m-k}}{2k-1}\,a^{2m-2k+1}\int\frac{x^{2k-1}}{x^2+a^2}\,dx.$$

**7.** $\displaystyle\int\frac{x^{2m+1}}{x^2+a^2}\operatorname{arctg}\frac{x}{a}\,dx = \frac{1}{2}\operatorname{arctg}\frac{x}{a}\sum_{k=1}^{m}\frac{(-1)^{m-k}}{k}\,a^{2m-2k}x^{2k} -$

$$- \frac{1}{2}\sum_{k=1}^{m}\frac{(-1)^{m-k}}{k}\,a^{2m-2k-1}\int\frac{x^{2k}\,dx}{x^2+a^2} + (-1)^m2^m\int\frac{x}{x^2+a^2}\operatorname{arctg}\frac{x}{a}\,dx.$$

**8.** $\displaystyle\int\frac{1}{x^2+a^2}\operatorname{arctg}\frac{x}{a}\,dx = \frac{1}{2a}\operatorname{arctg}^2\frac{x}{a}.$

**9.** $\displaystyle\int\frac{x^2}{x^2+a^2}\operatorname{arctg}\frac{x}{a}\,dx = x\operatorname{arctg}\frac{x}{a} - \frac{a}{2}\ln(x^2+a^2) - \frac{a}{2}\operatorname{arctg}^2\frac{x}{a}.$

**10.** $$\int \frac{1}{(x^2+a^2)^2} \operatorname{arctg}\frac{x}{a}\,dx = \frac{x}{2a^2(x^2+a^2)}\operatorname{arctg}\frac{x}{a} + \frac{1}{4a^3}\operatorname{arctg}^2\frac{x}{a} + \frac{1}{4a(x^2+a^2)}.$$

**11.** $$\int \frac{1}{x^2+a^2}\operatorname{arctg}^r\frac{x}{a}\,dx = \frac{1}{(r+1)a}\operatorname{arctg}^{r+1}\frac{x}{a}.$$

**12.** $$\int \frac{dx}{(x^2+a^2)\operatorname{arctg}\frac{x}{a}}\,dx = \frac{1}{a}\ln\left|\operatorname{arctg}\frac{x}{a}\right|.$$

**13.** $$\int \frac{x}{\sqrt{a^2-x^2}}\operatorname{arctg}\frac{x}{a}\,dx = -\sqrt{a^2-x^2}\operatorname{arctg}\frac{x}{a} - a\arcsin\frac{x}{a} + \sqrt{2}\,a\operatorname{arctg}\frac{\sqrt{2}\,x}{\sqrt{a^2-x^2}}.$$

**14.** $$\int \frac{\operatorname{arctg} x}{(ax^2+b)^{3/2}}\,dx = \frac{x\operatorname{arctg} x}{b\sqrt{ax^2+b}} - \frac{1}{b\sqrt{a-b}}\operatorname{arctg}\sqrt{\frac{ax^2+b}{a-b}} \qquad [a>b].$$

**15.** $$= \frac{x\operatorname{arctg} x}{b\sqrt{ax^2+b}} - \frac{1}{2b\sqrt{b-a}}\ln\frac{\sqrt{ax^2+b}-\sqrt{b-a}}{\sqrt{ax^2+b}+\sqrt{b-a}} \qquad [b>a].$$

### 7.14. Интегралы вида $\int x^p \operatorname{arcctg}\frac{x}{a}\,dx$.

**1.** $$\int x^p\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^{p+1}}{p+1}\operatorname{arcctg}\frac{x}{a} + \frac{a}{p+1}\int\frac{x^{p+1}\,dx}{x^2+a^2}.$$

**2.** $$\int x^{2n}\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^{2n+1}}{2n+1}\operatorname{arcctg}\frac{x}{a} + \frac{a}{2(2n+1)}\left[n!\sum_{k=0}^{n-1}\frac{(-a^2)^k(x^2+a^2)^{n-k}}{(n-k)!k!(n-k)} + (-a^2)^n\ln(x^2+a^2)\right].$$

**3.** $$\int x^{2n+1}\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^{2n+2}}{2n+2}\operatorname{arcctg}\frac{x}{a} + \frac{a^{2n+2}}{2n+2}\left[\sum_{k=1}^{n+1}\frac{(-1)^{n+k+1}}{2k-1}\left(\frac{x}{a}\right)^{2k-1} + (-1)^n\operatorname{arcctg}\frac{x}{a}\right].$$

**4.** $$\int \operatorname{arcctg}\frac{x}{a}\,dx = x\operatorname{arcctg}\frac{x}{a} + \frac{a}{2}\ln(x^2+a^2).$$

**5.** $$\int x\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^2+a^2}{2}\operatorname{arcctg}\frac{x}{a} + \frac{ax}{2}.$$

**6.** $$\int x^2\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^3}{3}\operatorname{arcctg}\frac{x}{a} + \frac{ax^2}{6} - \frac{a^3}{6}\ln(x^2+a^2).$$

**7.** $$\int x^3\operatorname{arcctg}\frac{x}{a}\,dx = \frac{x^4-a^4}{4}\operatorname{arcctg}\frac{x}{a} + \frac{ax^3}{12} - \frac{a^3x}{4}.$$

### 7.15. Интегралы вида $\int \frac{1}{x^p}\operatorname{arcctg}\frac{x}{a}\,dx$.

**1.** $$\int \frac{1}{x^p}\operatorname{arcctg}\frac{x}{a}\,dx = -\frac{1}{(p-1)x^{p-1}}\operatorname{arcctg}\frac{x}{a} - \frac{a}{p-1}\int\frac{dx}{x^{p-1}(x^2+a^2)}.$$

**2.** $$\int \frac{1}{x^{2n}} \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{1}{(2n-1)x^{2n-1}} \operatorname{arcctg} \frac{x}{a} + \frac{1}{2(2n-1)a^{2n-1}} \left[ \sum_{k=0}^{n-2} \frac{(-1)^{n-k}}{k+1} \left(\frac{a}{x}\right)^{2k+2} + (-1)^{n-1} \ln \frac{x^2+a^2}{x^2} \right].$$

**3.** $$\int \frac{1}{x^{2n+1}} \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{1}{2nx^{2n}} \operatorname{arcctg} \frac{x}{a} - \frac{1}{2na^{2n}} \left[ \sum_{k=1}^{n} \frac{(-1)^{k}}{2n-2k+1} \left(\frac{a}{x}\right)^{2n-2k+1} + (-1)^{n} \operatorname{arcctg} \frac{x}{a} \right].$$

**4.** $$\int \frac{1}{x} \operatorname{arcctg} \frac{x}{a}\, dx = \frac{\pi}{2} \ln|x| - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{x}{a}\right)^{2k+1} \qquad [|x| < |a|].$$

**5.** $$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x}\right)^{2k+1} \qquad [x/a > 1].$$

**6.** $$= \pi \ln|x| - \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left(\frac{a}{x}\right)^{2k+1} \qquad [x/a < -1].$$

**7.** $$\int \frac{1}{x^2} \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{1}{x} \operatorname{arcctg} \frac{x}{a} - \frac{1}{2a} \ln \frac{x^2}{x^2+a^2}.$$

**8.** $$\int \frac{1}{x^3} \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{1}{2}\left(\frac{1}{x^2} + \frac{1}{a^2}\right) \operatorname{arcctg} \frac{x}{a} + \frac{1}{2ax}.$$

**9.** $$\int \frac{1}{(x+b)^2} \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{a}{a^2+b^2} \left[ \ln \frac{x+b}{\sqrt{x^2+a^2}} - \frac{a^2-bx}{a(x+b)} \operatorname{arcctg} \frac{x}{a} \right] - \frac{\pi}{2(x+b)}.$$

### 7.16. Интегралы вида $\int x^p (x^2+a^2)^q \operatorname{arcctg} \frac{x}{a}\, dx$.

**1.** $$\int x^p (x^2+a^2)^n \operatorname{arcctg} \frac{x}{a}\, dx = \operatorname{arcctg} \frac{x}{a} \sum_{k=0}^{n} C_n^k \frac{a^{2n-2k} x^{p+2k+1}}{p+2k+1} + \sum_{k=0}^{n} C_n^k \frac{a^{2n-2k+1}}{p+2k+1} \int \frac{x^{p+2k+1}}{x^2+a^2}\, dx.$$

Если $k = -\dfrac{p+1}{2}$ $(k = 0, 1, \ldots, n)$, то соответствующие члены сумм заменяются на $C_n^k a^{p+2n+1} \ln \dfrac{x}{a} \operatorname{arcctg} \dfrac{x}{a}$ и $C_n^k a^{p+2n+2} \displaystyle\int \frac{1}{x^2+a^2} \ln \frac{x}{a}\, dx$.

**2.** $$\int \frac{x^p}{(x^2+a^2)^q} \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{x^{p-1}}{2(q-1)(x^2+a^2)^{q-1}} \operatorname{arcctg} \frac{x}{a} - \frac{a}{2(q-1)} \int \frac{x^{p-1}\, dx}{(x^2+a^2)^q} + \frac{p-1}{2(q-1)} \int \frac{x^{p-2}}{(x^2+a^2)^{q-1}} \operatorname{arcctg} \frac{x}{a}\, dx.$$

**3.** $$\int \frac{1}{(x^2+a^2)^q} \operatorname{arcctg} \frac{x}{a}\, dx = \frac{x}{2(q-1)a^2(x^2+a^2)^{q-1}} \operatorname{arcctg} \frac{x}{a} - \frac{1}{4(q-1)^2 a (x^2+a^2)^{q-1}} + \frac{2q-3}{2(q-1)a^2} \int \frac{1}{(x^2+a^2)^{q-1}} \operatorname{arcctg} \frac{x}{a}\, dx.$$

**4.** $$\int \frac{x^p}{x^2+a^2} \operatorname{arcctg} \frac{x}{a}\, dx = \int x^{p-2} \operatorname{arcctg} \frac{x}{a}\, dx - a^2 \int \frac{x^{p-2}}{x^2+a^2} \operatorname{arcctg} \frac{x}{a}\, dx.$$

**5.** $$\int \frac{x^{2m}}{x^2+a^2} \operatorname{arcctg} \frac{x}{a}\, dx = \frac{(-1)^{m+1}}{2}\, a^{2m-1} \operatorname{arcctg}^2 \frac{x}{a} +$$
$$+ \operatorname{arcctg} \frac{x}{a} \sum_{k=1}^{m} \frac{(-1)^{m-k}}{2k-1}\, a^{2m-2k} x^{2k-1} + \sum_{k=1}^{m} \frac{(-1)^{m-k}}{2k-1}\, a^{2m-2k+1} \int \frac{x^{2k-1}}{x^2+a^2}\, dx.$$

**6.** $$\int \frac{x^{2m+1}}{x^2+a^2} \operatorname{arcctg} \frac{x}{a}\, dx = \frac{1}{2} \operatorname{arcctg} \frac{x}{a} \sum_{k=1}^{m} \frac{(-1)^{m-k}}{k}\, a^{2m-2k} x^{2k} +$$
$$+ \frac{1}{2} \sum_{k=1}^{m} \frac{(-1)^{m-k}}{k}\, a^{2m-2k-1} \int \frac{x^{2k}\, dx}{x^2+a^2} + (-1)^m 2^m \int \frac{x}{x^2+a^2} \operatorname{arcctg} \frac{x}{a}\, dx.$$

**7.** $$\int \frac{1}{x^2+a^2} \operatorname{arcctg} \frac{x}{a}\, dx = -\frac{1}{2a} \operatorname{arcctg}^2 \frac{x}{a}.$$

**8.** $$\int \frac{x^2}{x^2+a^2} \operatorname{arcctg} \frac{x}{a}\, dx = x \operatorname{arcctg} \frac{x}{a} + \frac{a}{2} \ln\left(x^2+a^2\right) + \frac{a}{2} \operatorname{arcctg}^2 \frac{x}{a}.$$

**9.** $$\int \frac{1}{(x^2+a^2)^2} \operatorname{arcctg} \frac{x}{a}\, dx =$$
$$= \frac{x}{2a^2(x^2+a^2)} \operatorname{arcctg} \frac{x}{a} - \frac{1}{4a^3} \operatorname{arcctg}^2 \frac{x}{a} - \frac{1}{4a(x^2+a^2)}.$$

**10.** $$\int \frac{1}{x^2+a^2} \operatorname{arcctg}^r \frac{x}{a}\, dx = -\frac{1}{(r+1)a} \operatorname{arcctg}^{r+1} \frac{x}{a}.$$

**11.** $$\int \frac{dx}{(x^2+a^2) \operatorname{arcctg}(x/a)} = -\frac{1}{a} \ln \left|\operatorname{arcctg} \frac{x}{a}\right|.$$

**12.** $$\int \frac{x}{\sqrt{a^2-x^2}} \operatorname{arcctg} \frac{x}{a}\, dx =$$
$$= -\sqrt{a^2-x^2} \operatorname{arcctg} \frac{x}{a} + a \arcsin \frac{x}{a} + \sqrt{2}\, a \operatorname{arcctg} \frac{\sqrt{2}\, x}{\sqrt{a^2-x^2}}.$$

## 8. ОБРАТНЫЕ ГИПЕРБОЛИЧЕСКИЕ ФУНКЦИИ

Неопределенные интегралы от обратных гиперболических функций вычисляются по формулам раздела 7 «Логарифмические функции» после замены

$$\operatorname{Arsh} x = \ln\left(x + \sqrt{x^2+1}\,\right),$$

$$\operatorname{Arch} x = \pm\ln\left(x + \sqrt{x^2-1}\,\right) \qquad [x \geqslant 1],$$

$$\operatorname{Arth} x = \frac{1}{2}\ln\frac{1+x}{1-x} \qquad [|x| < 1],$$

$$\operatorname{Arcth} x = \frac{1}{2}\ln\frac{x+1}{x-1} \qquad [|x| > 1],$$

$$\operatorname{Arsech} x = \pm\ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\,\right) \qquad [0 < x \leqslant 1],$$

$$\operatorname{Arcosech} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2}+1}\,\right) \qquad [x > 0].$$

# Приложение I. Некоторые элементарные функции и их свойства

## I.1. Степенная, показательная и логарифмическая функции

**1.** Формулы дифференцирования.

$$\frac{d}{dx}\,x^p = px^{p-1}, \quad \frac{d}{dx}e^{ax} = ae^{ax}, \quad \frac{d}{dx}\ln|x| = \frac{1}{x},$$

$$\frac{d}{dx}\ln\left(x + \sqrt{x^2+a^2}\,\right) = \frac{1}{\sqrt{x^2+a^2}}, \quad \frac{d}{dx}\ln\left|x + \sqrt{x^2-a^2}\,\right| = \frac{1}{\sqrt{x^2-a^2}},$$

$$\frac{d}{dx}\ln\left|\frac{a+x}{a-x}\right| = \frac{2a}{a^2-x^2}.$$

**2.** Степенные ряды.

$$(1+x)^n = \sum_{k=0}^{n} C_n^k x^k \qquad [n = 0,\, 1,\, 2,\, \ldots],$$

$$(1-x)^n = \sum_{k=0}^{n} (-1)^k C_n^k x^k \qquad [n = 0,\, 1,\, 2,\, \ldots],$$

$$(1+x)^p = 1 + \sum_{k=1}^{\infty} \frac{p(p+1)\ldots(p+k-1)}{k!}\,x^k$$

$$[|x| < 1 \text{ при } p < 0,\ |x| \leqslant 1 \text{ при } p > 0].$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!},$$

$$\ln(1-x) = -\sum_{k=1}^{\infty} \frac{x^k}{k} \qquad [-1 \leqslant x < 1],$$

$$\ln\frac{1+x}{1-x} = 2\sum_{k=0}^{\infty} \frac{x^{2k+1}}{2k+1} \qquad [|x| < 1],$$

$$\ln\left(x + \sqrt{x^2+1}\,\right) = x + \sum_{k=1}^{\infty} (-1)^k \frac{1\cdot 3\ldots(2k-1)}{2\cdot 4\ldots(2k)}\,\frac{x^{2k+1}}{2k+1} \qquad [|x| < 1],$$

$$= \ln 2x + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1\cdot 3\ldots(2k-1)}{2\cdot 4\ldots(2k)}\,\frac{x^{-2k}}{2k} \qquad [x > 1],$$

$$= -\ln|2x| + \sum_{k=1}^{\infty} (-1)^k \frac{1\cdot 3\ldots(2k-1)}{2\cdot 4\ldots(2k)}\,\frac{x^{-2k}}{2k} \qquad [x < -1],$$

$$\ln\left(x + \sqrt{x^2-1}\,\right) = \ln 2x - \sum_{k=1}^{\infty} \frac{1\cdot 3\ldots(2k-1)}{2\cdot 4\ldots(2k)}\,\frac{x^{-2k}}{2k} \qquad [x > 1],$$

$$\ln\left(\sqrt{x^2+1} - x\right) = -\ln\left(x + \sqrt{x^2+1}\,\right), \quad \ln\left(x - \sqrt{x^2-1}\,\right) = -\ln\left(x + \sqrt{x^2-1}\,\right).$$

## I.2. Гиперболические функции

**1.** Некоторые соотношения.

$$\operatorname{sh} x = -\operatorname{sh}(-x) = \pm\sqrt{\operatorname{ch}^2 x - 1} = \pm\sqrt{\frac{1}{2}(\operatorname{ch} 2x - 1)} = \frac{e^x - e^{-x}}{2},$$

$$\operatorname{ch} x = \operatorname{ch}(-x) = \sqrt{\operatorname{sh}^2 x + 1} = \sqrt{\frac{1}{2}(\operatorname{ch} 2x + 1)} = 2\operatorname{ch}^2\frac{x}{2} - 1 = \frac{e^x + e^{-x}}{2},$$

$$\operatorname{th} x = -\operatorname{th}(-x) = \frac{\operatorname{sh} x}{\operatorname{ch} x} = \frac{\operatorname{sh} 2x}{\operatorname{ch} 2x + 1} = \frac{\operatorname{ch} 2x - 1}{\operatorname{sh} 2x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\operatorname{cth} x = -\operatorname{cth}(-x) = \frac{\operatorname{ch} x}{\operatorname{sh} x} = \frac{\operatorname{sh} 2x}{\operatorname{ch} 2x - 1} = \frac{\operatorname{ch} 2x + 1}{\operatorname{ch} 2x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{1}{\operatorname{ch} x}, \quad \operatorname{cosech} x = \frac{1}{\operatorname{sh} x},$$

$$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1, \quad \operatorname{th}^2 x + \operatorname{sech}^2 x = 1, \quad \operatorname{cth}^2 x - \operatorname{cosech}^2 x = 1,$$

$$(\operatorname{sh} x + \operatorname{ch} x)^n = \operatorname{sh} nx + \operatorname{ch} nx \qquad [n = 0,\ \pm 1,\ \pm 2,\ \ldots].$$

**2.** Формулы дифференцирования.

$$\frac{d\operatorname{sh} x}{dx} = \operatorname{ch} x, \quad \frac{d\operatorname{cosech} x}{dx} = -\frac{\operatorname{ch} x}{\operatorname{sh}^2 x}, \quad \frac{d\operatorname{ch} x}{dx} = \operatorname{sh} x,$$

$$\frac{d\operatorname{th} x}{dx} = \operatorname{sech}^2 x, \quad \frac{d\operatorname{sech} x}{dx} = -\frac{\operatorname{sh} x}{\operatorname{ch}^2 x}, \quad \frac{d\operatorname{cth} x}{dx} = -\operatorname{cosech}^2 x.$$

**3.** Связь с тригонометрическими функциями.

$$\operatorname{sh}(ix) = i\sin x, \quad \operatorname{ch}(ix) = \cos x, \quad \operatorname{th}(ix) = i\operatorname{tg} x, \quad \operatorname{cth}(ix) = -i\operatorname{ctg} x.$$

**4.** Гиперболические функции суммы и разности углов.

$$\operatorname{sh}(x \pm y) = \operatorname{sh} x\operatorname{ch} y \pm \operatorname{sh} y\operatorname{ch} x, \quad \operatorname{ch}(x \pm y) = \operatorname{ch} x\operatorname{ch} y \pm \operatorname{sh} x\operatorname{sh} y,$$

$$\operatorname{sh}(x \pm iy) = \operatorname{sh} x\cos y \pm i\sin y\operatorname{ch} x, \quad \operatorname{ch}(x \pm iy) = \operatorname{ch} x\cos y \pm i\operatorname{sh} x\sin y,$$

$$\operatorname{th}(x \pm y) = \frac{\operatorname{th} x \pm \operatorname{th} y}{1 \pm \operatorname{th} x\operatorname{th} y} = \frac{\operatorname{sh} 2x \pm \operatorname{sh} 2y}{\operatorname{ch} 2x + \operatorname{ch} 2y},$$

$$\operatorname{th}(x \pm iy) = \frac{\operatorname{th} x \pm i\operatorname{tg} y}{1 \pm i\operatorname{th} x\operatorname{tg} y} = \frac{\operatorname{sh} 2x \pm i\sin 2y}{\operatorname{ch} 2x + \cos 2y},$$

$$\operatorname{cth}(x \pm iy) = \frac{\operatorname{sh} 2x \mp i\sin 2y}{\operatorname{ch} 2x - \cos 2y}.$$

**5.** Суммы и разности гиперболических функций.

$$\operatorname{sh} x \pm \operatorname{sh} y = 2\operatorname{sh}\frac{x \pm y}{2}\operatorname{ch}\frac{x \mp y}{2}, \quad \operatorname{ch} x + \operatorname{ch} y = 2\operatorname{ch}\frac{x + y}{2}\operatorname{ch}\frac{x - y}{2},$$

$$\operatorname{ch} x - \operatorname{ch} y = 2\operatorname{sh}\frac{x + y}{2}\operatorname{sh}\frac{x - y}{2},$$

$$\operatorname{th} x \pm \operatorname{th} y = \frac{\operatorname{sh}(x \pm y)}{\operatorname{ch} x\operatorname{ch} y}, \quad \operatorname{cth} x \pm \operatorname{cth} y = \pm\frac{\operatorname{sh}(x \pm y)}{\operatorname{sh} x\operatorname{sh} y},$$

$$\operatorname{sh}^2 x - \operatorname{sh}^2 y = \operatorname{ch}^2 x - \operatorname{ch}^2 y = \operatorname{sh}(x + y)\operatorname{sh}(x - y),$$

$$\operatorname{sh}^2 x + \operatorname{ch}^2 y = \operatorname{ch}(x + y)\operatorname{ch}(x - y).$$

**6.** Произведения гиперболических функций.

$$\operatorname{sh} x \operatorname{sh} y = \frac{1}{2}\left[\operatorname{ch}(x+y) - \operatorname{ch}(x-y)\right], \quad \operatorname{ch} x \operatorname{ch} y = \frac{1}{2}\left[\operatorname{ch}(x+y) + \operatorname{ch}(x-y)\right],$$

$$\operatorname{sh} x \operatorname{ch} y = \frac{1}{2}\left[\operatorname{sh}(x+y) + \operatorname{sh}(x-y)\right].$$

**7.** Степени гиперболических функций.

$$\operatorname{sh}^{2n} x = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} (-1)^k C_{2n}^k \operatorname{sh} 2(n-k)x + \frac{(-1)^n}{2^{2n}} C_{2n}^n,$$

$$\operatorname{sh}^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} (-1)^k C_{2n+1}^k \operatorname{sh}(2n-2k+1)x,$$

$$\operatorname{sh}^2 x = \frac{1}{2} \operatorname{ch} 2x - \frac{1}{2}, \quad \operatorname{sh}^3 x = \frac{1}{4} \operatorname{sh} 3x - \frac{3}{4} \operatorname{sh} x,$$

$$\operatorname{sh}^4 x = \frac{1}{8} \operatorname{ch} 4x - \frac{1}{2} \operatorname{ch} 2x + \frac{3}{8},$$

$$\operatorname{sh}^5 x = \frac{1}{16} \operatorname{sh} 5x - \frac{5}{16} \operatorname{sh} 3x + \frac{5}{8} \operatorname{sh} x,$$

$$\operatorname{sh}^6 x = \frac{1}{32} \operatorname{ch} 6x - \frac{3}{16} \operatorname{ch} 4x + \frac{15}{32} \operatorname{ch} 2x - \frac{5}{16},$$

$$\operatorname{ch}^{2n} x = \frac{1}{2^{2n-1}} \sum_{k=0}^{n-1} C_{2n}^k \operatorname{ch} 2(n-k)x + \frac{1}{2^{2n}} C_{2n}^n,$$

$$\operatorname{ch}^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} C_{2n+1}^k \operatorname{ch}(2n-2k+1)x,$$

$$\operatorname{ch}^2 x = \frac{1}{2} \operatorname{ch} 2x + \frac{1}{2}, \quad \operatorname{ch}^3 x = \frac{1}{4} \operatorname{ch} 3x + \frac{3}{4} \operatorname{ch} x,$$

$$\operatorname{ch}^4 x = \frac{1}{8} \operatorname{ch} 4x + \frac{1}{2} \operatorname{ch} 2x + \frac{3}{8},$$

$$\operatorname{ch}^5 x = \frac{1}{16} \operatorname{ch} 5x + \frac{5}{16} \operatorname{ch} 3x + \frac{5}{8} \operatorname{ch} x,$$

$$\operatorname{ch}^6 x = \frac{1}{32} \operatorname{ch} 6x + \frac{3}{16} \operatorname{ch} 4x + \frac{15}{32} \operatorname{ch} 2x + \frac{5}{16}.$$

**8.** Гиперболические функции кратных аргументов.

$$\begin{aligned} \operatorname{sh} nx &= \sum_{k=0}^{[(n-1)/2]} C_n^{2k+1} \operatorname{sh}^{2k+1} x \operatorname{ch}^{n-2k-1} x, \\ &= \operatorname{sh} x \sum_{k=0}^{[(n-1)/2]} (-1)^k 2^{n-2k-1} C_{n-k-1}^k \operatorname{ch}^{n-2k-1} x, \end{aligned}$$

$$\begin{aligned} \operatorname{sh}(2n+1)x &= \operatorname{sh} x \left(1 + 2\sum_{k=1}^{n} \operatorname{ch} 2kx\right), \\ &= (-1)^n \left(\operatorname{sh} x + 2 \operatorname{ch} x \sum_{k=1}^{n} (-1)^k \operatorname{sh} 2kx\right), \end{aligned}$$

$$\operatorname{sh} 2x = 2 \operatorname{sh} x \operatorname{ch} x, \quad \operatorname{sh} 3x = 4 \operatorname{sh}^3 x + 3 \operatorname{sh} x,$$

$$\operatorname{sh} 4x = \operatorname{ch} x (4 \operatorname{sh} x + 8 \operatorname{sh}^3 x), \quad \operatorname{sh} 5x = 16 \operatorname{sh}^5 x + 20 \operatorname{sh}^3 x + 5 \operatorname{sh} x,$$

$$\operatorname{ch} nx = \sum_{k=0}^{[n/2]} C_n^{2k} \operatorname{sh}^{2k} x \operatorname{ch}^{n-2k} x,$$

$$= 2^{n-1} \operatorname{ch}^n x + \frac{n}{2} \sum_{k=0}^{[n/2]-1} \frac{(-1)^{k+1}}{k+1} C_{n-k-2}^{k} 2^{n-2k-2} \operatorname{ch}^{n-2k-2} x,$$

$$\operatorname{ch}(2n+1)x = \operatorname{ch} x + 2 \operatorname{sh} x \sum_{k=1}^{n} \operatorname{sh} 2kx,$$

$$= (-1)^n \operatorname{ch} x \left(1 + 2 \sum_{k=1}^{n} (-1)^k \operatorname{ch} 2kx\right),$$

$$\operatorname{ch} 2x = 2 \operatorname{ch}^2 x - 1, \quad \operatorname{ch} 3x = 4 \operatorname{ch}^3 x - 3 \operatorname{ch} x,$$

$$\operatorname{ch} 4x = 8 \operatorname{ch}^4 x - 8 \operatorname{ch}^2 x + 1, \quad \operatorname{ch} 5x = 16 \operatorname{ch}^5 x - 20 \operatorname{ch}^3 x + 5 \operatorname{ch} x.$$

**9.** Степенные ряды.

$$\operatorname{sh} x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}, \quad \operatorname{ch} x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!},$$

$$\frac{1}{\operatorname{sh} x} = \frac{1}{x} - 2 \sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{(2k)!} B_{2k} x^{2k-1} \qquad [|x| < \pi],$$

$$\frac{1}{\operatorname{ch} x} = 1 + \sum_{k=1}^{\infty} \frac{E_{2k}}{(2k)!} x^{2k} \qquad [|x| < \pi/2],$$

$$\operatorname{th} x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!} B_{2k} x^{2k-1} \qquad [|x| < \pi/2],$$

$$\operatorname{cth} x = \frac{1}{x} + \sum_{k=1}^{\infty} \frac{2^{2k}}{(2k)!} B_{2k} x^{2k-1} \qquad [|x| < \pi].$$

## I.3. Тригонометрические функции

**1.** Некоторые соотношения.

$$\sin x = -\sin(-x) = \pm\sqrt{1-\cos^2 x} = \pm\sqrt{\frac{1}{2}(1-\cos 2x)} =$$

$$= 2\cos^2\left(\frac{x}{2} - \frac{\pi}{4}\right) - 1 = \frac{2 \operatorname{tg}(x/2)}{1+\operatorname{tg}^2(x/2)} = \frac{e^{ix}-e^{-ix}}{2i},$$

$$\cos x = \cos(-x) = \pm\sqrt{1-\sin^2 x} = \pm\sqrt{\frac{1}{2}(1+\cos 2x)} =$$

$$= 2\cos^2 \frac{x}{2} - 1 = \frac{1-\operatorname{tg}^2(x/2)}{1+\operatorname{tg}^2(x/2)} = \frac{e^{ix}+e^{-ix}}{2},$$

$$\operatorname{tg} x = -\operatorname{tg}(-x) = \frac{\sin x}{\cos x} = \pm\sqrt{\sec^2 x - 1} = \frac{\sin 2x}{1+\cos 2x} =$$

$$= \frac{1-\cos 2x}{\sin 2x} = \pm\sqrt{\frac{1-\cos 2x}{1+\cos 2x}} = \frac{2\operatorname{tg}(x/2)}{1-\operatorname{tg}^2(x/2)} = -i\frac{e^{ix}-e^{-ix}}{e^{ix}+e^{-ix}},$$

$$\operatorname{ctg} x = -\operatorname{ctg}(-x) = \frac{\cos x}{\sin x} = \pm\sqrt{\operatorname{cosec}^2 x - 1} = \frac{1+\cos 2x}{\sin 2x} =$$

$$= \frac{\sin 2x}{1-\cos 2x} = \pm\sqrt{\frac{1+\cos 2x}{1-\cos 2x}} = \frac{1-\operatorname{tg}^2(x/2)}{2\operatorname{tg}(x/2)} = i\frac{e^{ix}+e^{-ix}}{e^{ix}-e^{-ix}},$$

$$\sec x = \sec(-x) = \frac{1}{\cos x} = \pm\sqrt{1+\operatorname{tg}^2 x} = \frac{1+\operatorname{tg}^2(x/2)}{1-\operatorname{tg}^2(x/2)},$$

$$\operatorname{cosec} x = -\operatorname{cosec}(-x) = \frac{1}{\sin x} = \pm\sqrt{1+\operatorname{ctg}^2 x} = \frac{1+\operatorname{tg}^2(x/2)}{2\operatorname{tg}(x/2)},$$

$$\sin^2 x + \cos^2 x = 1, \qquad \sec^2 x - \operatorname{tg}^2 x = 1, \qquad \operatorname{cosec}^2 x - \operatorname{ctg}^2 x = 1.$$

**2.** Формулы дифференцирования.

$$\frac{d\sin x}{dx} = \cos x, \qquad \frac{d\operatorname{tg} x}{dx} = \sec^2 x, \qquad \frac{d\sec x}{dx} = \frac{\sin x}{\cos^2 x},$$

$$\frac{d\cos x}{dx} = -\sin x, \qquad \frac{d\operatorname{ctg} x}{dx} = -\operatorname{cosec}^2 x, \qquad \frac{d\operatorname{cosec} x}{dx} = -\frac{\cos x}{\sin^2 x}.$$

**3.** Формулы Эйлера и Муавра.

$$e^{z+ix} = e^z(\cos x + i\sin x),$$

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx \qquad [n = 0,\ \pm1,\ \pm2,\ \ldots].$$

**4.** Связь с гиперболическими функциями.

$$\sin(ix) = i\operatorname{sh} x, \qquad \cos(ix) = \operatorname{ch} x,$$

$$\operatorname{tg}(ix) = i\operatorname{th} x, \qquad \operatorname{ctg}(ix) = -i\operatorname{cth} x,$$

$$\sec(ix) = \operatorname{sech} x, \qquad \operatorname{cosec}(ix) = -i\operatorname{cosech} x.$$

**5.** Тригонометрические функции суммы и разности.

$$\sin(x\pm y) = \sin x\cos y \pm \sin y\cos x,$$

$$\cos(x\pm y) = \cos x\cos y \mp \sin x\sin y,$$

$$\sin(x\pm iy) = \sin x\operatorname{ch} y \pm i\operatorname{sh} y\cos x,$$

$$\cos(x\pm iy) = \cos x\operatorname{ch} y \mp i\sin x\operatorname{sh} y,$$

$$\operatorname{tg}(x\pm y) = \frac{\operatorname{tg} x \pm \operatorname{tg} y}{1\mp\operatorname{tg} x\operatorname{tg} y}, \qquad \operatorname{ctg}(x\pm y) = \frac{1\mp\operatorname{tg} x\operatorname{tg} y}{\operatorname{tg} x\pm\operatorname{tg} y},$$

$$\operatorname{tg}(x\pm iy) = \frac{\operatorname{tg} x \pm i\operatorname{th} y}{1\mp i\operatorname{tg} x\operatorname{th} y} = \frac{\sin 2x \pm i\operatorname{sh} 2y}{\cos 2x + \operatorname{ch} 2y},$$

$$\operatorname{ctg}(x\pm iy) = \frac{1\pm i\operatorname{ctg} x\operatorname{cth} y}{-\operatorname{ctg} x \pm i\operatorname{cth} y} = \frac{\sin 2x \mp i\operatorname{sh} 2y}{\operatorname{ch} 2y - \cos 2x}.$$

**6.** Формулы приведения.

$$\sin(x\pm n\pi) = (-1)^n\sin x, \qquad \cos(x\pm n\pi) = (-1)^n\cos x,$$

$$\operatorname{tg}(x\pm n\pi) = \operatorname{tg} x, \qquad \operatorname{ctg}(x\pm n\pi) = \operatorname{ctg} x,$$

$$\sin\left(x\pm\frac{2n+1}{2}\pi\right) = \pm(-1)^n\cos x, \qquad \cos\left(x\pm\frac{2n+1}{2}\pi\right) = \mp(-1)^n\sin x,$$

$$\operatorname{tg}\left(x\pm\frac{2n+1}{2}\pi\right) = -\operatorname{ctg} x, \qquad \operatorname{ctg}\left(x\pm\frac{2n+1}{2}\pi\right) = -\operatorname{tg} x,$$

$$\sin\left(x\pm\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\sin x\pm\cos x), \qquad \cos\left(x\pm\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}(\cos x\mp\sin x),$$

$$\operatorname{tg}\left(x\pm\frac{\pi}{4}\right) = \frac{\sin x\pm\cos x}{\cos x\mp\sin x} = \frac{\operatorname{tg} x\pm 1}{1\mp\operatorname{tg} x},$$

$$\operatorname{ctg}\left(x \pm \frac{\pi}{4}\right) = \frac{\cos x \mp \sin x}{\sin x \pm \cos x} = \frac{\operatorname{ctg} x \mp 1}{1 \pm \operatorname{ctg} x}.$$

**7.** Суммы и разности тригонометрических функций.

$$\sin x \pm \sin y = 2\sin\frac{x \pm y}{2}\cos\frac{x \mp y}{2},$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2},$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2},$$

$$\operatorname{tg} x \pm \operatorname{tg} y = \frac{\sin(x \pm y)}{\cos x \cos y}, \quad \operatorname{ctg} x \pm \operatorname{ctg} y = -\frac{\sin(x \pm y)}{\sin x \sin y},$$

$$\operatorname{tg} x - \operatorname{ctg} y = -\frac{\cos(x+y)}{\cos x \sin y},$$

$$a\cos x + b\sin x = r\sin(x+\varphi) = r\cos(x-\psi),$$

где

$$r = \sqrt{a^2+b^2}, \quad \sin\varphi = a/r, \quad \cos\varphi = b/r, \quad \sin\psi = b/r, \quad \cos\psi = a/r,$$

$$\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x = \sin(x+y)\sin(x-y),$$

$$\sin^2 x - \cos^2 y = -\cos(x+y)\cos(x-y), \quad \sec^2 x + \operatorname{cosec}^2 x = \sec^2 x \operatorname{cosec}^2 x.$$

**8.** Произведения тригонометрических функций.

$$\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)],$$

$$\cos x \cos y = \frac{1}{2}[\cos(x-y) + \cos(x+y)],$$

$$\sin x \cos y = \frac{1}{2}[\sin(x-y) + \sin(x+y)].$$

**9.** Степени тригонометрических функций.

$$\sin^{2n} x = \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k C_{2n}^k \cos 2(n-k)x + \frac{1}{2^{2n}}C_{2n}^n,$$

$$\sin^{2n+1} x = \frac{(-1)^n}{2^{2n}}\sum_{k=0}^{n}(-1)^k C_{2n+1}^k \sin(2n-2k+1)x,$$

$$\sin^2 x = -\frac{1}{2}\cos 2x + \frac{1}{2}, \quad \sin^3 x = -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x,$$

$$\sin^4 x = \frac{1}{8}\cos 4x - \frac{1}{2}\cos 2x + \frac{3}{8},$$

$$\sin^5 x = \frac{1}{16}\sin 5x - \frac{5}{16}\sin 3x + \frac{5}{8}\sin x,$$

$$\sin^6 x = -\frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x - \frac{15}{32}\cos 2x + \frac{5}{16},$$

$$\cos^{2n} x = \frac{1}{2^{2n-1}}\sum_{k=0}^{n-1} C_{2n}^k \cos 2(n-k)x + \frac{1}{2^{2n}}C_{2n}^n,$$

$$\cos^{2n+1} x = \frac{1}{2^{2n}}\sum_{k=0}^{n} C_{2n+1}^k \sin(2n-2k+1)x,$$

$$\cos^2 x = \frac{1}{2}\cos 2x + \frac{1}{2}, \quad \cos^3 x = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x,$$

$$\cos^4 x = \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8},$$
$$\cos^5 x = \frac{1}{16}\cos 5x + \frac{5}{16}\cos 3x + \frac{5}{8}\cos x,$$
$$\cos^6 x = \frac{1}{32}\cos 6x + \frac{3}{16}\cos 4x + \frac{15}{32}\cos 2x + \frac{5}{16}.$$

**10.** Тригонометрические функции кратных аргументов.

$$\sin nx = \sum_{k=0}^{[(n-1)/2]} (-1)^k C_n^{2k+1} \sin^{2k+1} x \cos^{n-2k-1} x,$$
$$= \sin x \sum_{k=0}^{[(n-1)/2]} (-1)^k C_{n-k-1}^k 2^{n-2k-1} \cos^{n-2k-1} x,$$

$$\sin 2nx = 2n\cos x\Biggl[\sin x + \sum_{k=1}^{n-1}(-1)^k \frac{(n^2-1^2)(n^2-2^2)\ldots(n^2-k^2)}{(2k-1)!}4^k \sin^{2k-1} x\Biggr],$$
$$= (-1)^{n-1}\cos x\Biggl[2^{2n-1}\sin^{2n-1} x + \sum_{k=1}^{n-1}(-1)^k \frac{(2n-k-1)(2n-k-2)\ldots(2n-2k)}{k!}2^{2n-2k-1}\sin^{2n-2k-1} x\Biggr],$$
$$[n \geqslant 1],$$

$$\sin(2n+1)x = (2n+1)\Biggl[\sin x + \sum_{k=1}^{n}(-1)^k \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k+1)!} \times \sin^{2k+1} x\Biggr],$$
$$= (-1)^n\Biggl[2^{2n}\sin^{2n+1} x - 2^{2n-2}(2n+1)\sin^{2n-1} x + \frac{2n+1}{2} \times \sum_{k=2}^{n}(-1)^k \frac{(2n-k)(2n-k-1)\ldots(2n-2k+2)}{k!}2^{2n-2k+1}\sin^{2n-2k+1} x\Biggr],$$
$$= \sin x\left(1 + 2\sum_{k=1}^{n}\cos 2kx\right),$$
$$= (-1)^n\left[\sin x + 2\cos x\sum_{k=1}^{n}(-1)^k \sin 2kx\right],$$

$$\sin 2x = 2\sin x\cos x, \qquad \sin 3x = 3\sin x - 4\sin^3 x,$$
$$\sin 4x = \cos x\,(4\,\sin x - 8\,\sin^3 x), \qquad \sin 5x = 5\sin x - 20\sin^3 x + 16\sin^5 x,$$
$$\sin 6x = \cos x(6\,\sin x - 32\sin^3 x + 32\sin^5 x),$$

$$\cos nx = \sum_{k=0}^{[n/2]} (-1)^k C_n^{2k} \sin^{2k} x \cos^{n-2k} x,$$

$$= n \sum_{k=0}^{[(n-1)/2]} \frac{(-1)^k}{n-2k} \binom{n-k-1}{k} 2^{n-2k-1} \cos n - 2kx,$$

$$\cos 2nx = 1 + \sum_{k=1}^{n} (-1)^k \frac{n^2(n^2-1^2)\ldots[n^2-(k-1)^2]}{(2k)!} 4^k \sin^{2k} x,$$

$$\cos(2n+1)x = \cos x \Bigg[1 + \sum_{k=1}^{n} (-1)^k \times$$

$$\times \frac{[(2n+1)^2-1^2][(2n+1)^2-3^2]\ldots[(2n+1)^2-(2k-1)^2]}{(2k)!} \sin^{2k} x\Bigg],$$

$$= (-1)^n \cos x \Bigg[2^{2n} \sin^{2n} x +$$

$$+ \sum_{k=1}^{n} (-1)^k \frac{(2n-k)(2n-k-1)\ldots(2n-2k+1)}{k!} 2^{2n-2k} \sin^{2n-2k} x\Bigg],$$

$$= \cos x - 2\sin x \sum_{k=1}^{n} \sin 2kx,$$

$$= (-1)^n \cos x \left[1 + 2\sum_{k=1}^{n} (-1)^k \cos 2kx\right],$$

$$\cos 2x = 2\cos^2 x - 1, \quad \cos 3x = 4\cos^3 x - 3\cos x,$$

$$\cos 4x = 8\cos^4 x - 8\cos^2 x + 1, \quad \cos 5x = 16\cos^5 x - 20\cos^3 x + 5\cos x,$$

$$\cos 6x = 32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1,$$

$$\operatorname{tg} 2x = \frac{2\operatorname{tg} x}{1-\operatorname{tg}^2 x}, \quad \operatorname{tg} 3x = \frac{3\operatorname{tg} x - \operatorname{tg}^3 x}{1-3\operatorname{tg}^2 x}, \quad \operatorname{tg} 4x = \frac{4\operatorname{tg} x - 4\operatorname{tg}^3 x}{1-6\operatorname{tg}^2 x + \operatorname{tg}^4 x}.$$

**11.** Тригонометрические функции некоторых углов.
1) $\sin n\pi = 0,\ \cos n\pi = (-1)^n,\ \operatorname{tg} n\pi = 0.$

2) $\sin(n\pi/2) = (-1)^{[n/2]} \dfrac{1-(-1)^n}{2}, \quad \cos(n\pi/2) = (-1)^{[n/2]} \dfrac{1+(-1)^n}{2},$

$$\operatorname{tg}\frac{n\pi}{2} = \frac{1-(-1)^n}{1+(-1)^n}.$$

3) $\sin\dfrac{n\pi}{3} = (-1)^{[n/3]} \dfrac{\sqrt{3}}{4}\left[1-(-1)^{[(n+1)/3]}\right]$ [$n$ четное],

$$= (-1)^{[n/3]} \frac{\sqrt{3}}{4}\left[1+(-1)^{[(n+1)/3]}\right] \quad [n \text{ нечетное}],$$

$$\cos\frac{n\pi}{3} = (-1)^{[(n+1)/3]} \frac{3+(-1)^{[(n+1)/3]}}{4} \quad [n \text{ четное}],$$

$$= (-1)^{[(n+1)/3]} \frac{3-(-1)^{[(n+1)/3]}}{4} \quad [n \text{ нечетное}],$$

$$\operatorname{tg}\frac{n\pi}{3} = (-1)^{[(n+2)/3]}\,\frac{\sqrt{3}}{2}\left[1-(-1)^{[(n+1)/3]}\right] \qquad [n \text{ четное}],$$

$$= (-1)^{[n/3]}\,\frac{\sqrt{3}}{2}\left[1+(-1)^{[(n+1)/3]}\right] \qquad [n \text{ нечетное}].$$

4) $$\sin\frac{n\pi}{4} = (-1)^{[n/4]}\,\frac{1-(-1)^{[n/2]}}{2} \qquad [n \text{ четное}],$$

$$= (-1)^{[(n-1)/4]}\,\frac{1}{\sqrt{2}} \qquad [n \text{ нечетное}],$$

$$\cos\frac{n\pi}{4} = (-1)^{[n/4]}\,\frac{1+(-1)^{[n/2]}}{2} \qquad [n \text{ четное}],$$

$$= (-1)^{[(n+1)/4]}\,\frac{1}{\sqrt{2}} \qquad [n \text{ нечетное}],$$

$$\operatorname{tg}\frac{n\pi}{4} = (-1)^{[n/4]}\,\frac{1-(-1)^{[n/2]}}{1+(-1)^{[n/2]}} \qquad [n \text{ четное}],$$

$$= (-1)^{[(2n+1)/4]} \qquad [n \text{ нечетное}].$$

5) $$\sin\frac{n\pi}{5} = \frac{1}{4}(-1)^{[n/5]}\sqrt{10-(-1)^{[(2n-10[n/5]-1)/3]}2\sqrt{5}} \qquad [n \neq 0,\ 5,\ 10,\ \ldots\ ],$$

$$= 0 \qquad [n = 0,\ 5,\ 10,\ \ldots\ ],$$

$$\cos\frac{n\pi}{5} = \frac{1}{4}(-1)^{[(n+2)/5]}\left(\sqrt{5}+(-1)^{[(2n-10[n/5]-1)/3]}\right) \qquad [n \neq 0,\ 5,\ 10,\ \ldots\ ].$$

$$= (-1)^{[(n+2)/5]} \qquad [n = 0,\ 5,\ 10,\ \ldots\ ],$$

6) $$\sin\frac{n\pi}{6} = (-1)^{[n/6]}\,\frac{\sqrt{3}}{4}\left[1-(-1)^{[(n+1)/3]}\right] \qquad [n \text{ четное}],$$

$$= (-1)^{[n/6]}\,\frac{3-(-1)^{[(n+1)/3]}}{4} \qquad [n \text{ нечетное}],$$

$$\cos\frac{n\pi}{6} = (-1)^{[(n+2)/6]}\,\frac{3+(-1)^{[(n+1)/3]}}{4} \qquad [n \text{ четное}],$$

$$= (-1)^{[(n+1)/6]}\,\frac{\sqrt{3}}{4}\left[1+(-1)^{[(n+1)/3]}\right] \qquad [n \text{ нечетное}],$$

$$\operatorname{tg}\frac{n\pi}{6} = (-1)^{[n/3]}\,\frac{\sqrt{3}}{2}\left[1-(-1)^{[(n+1)/3]}\right] \qquad [n \text{ четное}],$$

$$= (-1)^{[n/3]}\,\frac{2}{\sqrt{3}\,[1+(-1)^{[(n+1)/3]}]} \qquad [n \text{ нечетное}].$$

7) $$\sin\frac{n\pi}{8} = (-1)^{[(n+1)/8]}\,\frac{1}{2}\sqrt{2-(-1)^{[n/4]}-(-1)^{[(3n+1)/4]}} \qquad [n \text{ четное}],$$

$$= (-1)^{[n/8]}\,\frac{1}{2}\sqrt{2-(-1)^{[(n+1)/4]}\sqrt{2}} \qquad [n \text{ нечетное}],$$

$$\cos\frac{n\pi}{8} = (-1)^{[(n+5)/8]}\,\frac{1}{2}\sqrt{2+(-1)^{[n/4]}+(-1)^{[(3n+1)/4]}} \qquad [n \text{ четное}],$$

$$= (-1)^{[(n+3)/8]}\,\frac{1}{2}\sqrt{2+(-1)^{[(n+1)/4]}\sqrt{2}} \qquad [n \text{ нечетное}],$$

$$\operatorname{tg}\frac{n\pi}{8} = (-1)^{[(n/2-4[n/8]-1)/2]} \qquad [n \neq 0,\ 4,\ 8,\ \dots\,],$$
$$= 0 \qquad [n = 0,\ 8,\ 16,\ \dots\,],$$
$$= (-1)^{[(2n+1)/8]}\left[\sqrt{2} - (-1)^{[(n+1)/4]}\right] \qquad [n \text{ нечетное}].$$

8) $$\sin\frac{\pi}{10} = \frac{1}{4}(\sqrt{5}-1),\ \sin\frac{3\pi}{10} = \frac{1}{4}(\sqrt{5}+1),$$
$$\cos\frac{\pi}{10} = \frac{1}{4}\sqrt{10+2\sqrt{5}},\ \cos\frac{3\pi}{10} = \frac{1}{4}\sqrt{10-2\sqrt{5}},$$
$$\operatorname{tg}\frac{\pi}{10} = \frac{1}{\sqrt{5}}\sqrt{5-2\sqrt{5}},\ \operatorname{tg}\frac{3\pi}{10} = \frac{1}{\sqrt{5}}\sqrt{5+2\sqrt{5}},$$
$$\operatorname{ctg}\frac{\pi}{10} = \sqrt{5+2\sqrt{5}},\ \operatorname{ctg}\frac{3\pi}{10} = \sqrt{5-2\sqrt{5}}.$$

9) $$\sin\frac{\pi}{12} = \frac{1}{4}(\sqrt{6}-\sqrt{2}), \quad \cos\frac{\pi}{12} = \frac{1}{4}(\sqrt{6}+\sqrt{2}),$$
$$\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}, \quad \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$
$$\operatorname{tg}\frac{\pi}{12} = 2-\sqrt{3}, \quad \operatorname{ctg}\frac{\pi}{12} = 2+\sqrt{3},$$
$$\operatorname{tg}\frac{5\pi}{12} = 2+\sqrt{3}, \quad \operatorname{ctg}\frac{5\pi}{12} = 2-\sqrt{3},$$
$$\sin\frac{7\pi}{12} = \frac{1}{2\sqrt{2}}(1+\sqrt{3}), \quad \cos\frac{7\pi}{12} = \frac{1}{2\sqrt{2}}(1-\sqrt{3}),$$
$$\sin\frac{11\pi}{12} = \frac{1}{2\sqrt{2}}(-1+\sqrt{3}), \quad \cos\frac{11\pi}{12} = \frac{1}{2\sqrt{2}}(-1-\sqrt{3}),$$
$$\operatorname{tg}\frac{7\pi}{12} = -2-\sqrt{3}, \quad \operatorname{ctg}\frac{7\pi}{12} = -2+\sqrt{3},$$
$$\operatorname{tg}\frac{11\pi}{12} = -2+\sqrt{3} \quad \operatorname{ctg}\frac{11\pi}{12} = -2-\sqrt{3}.$$

10) $$\sin\frac{\pi}{15} = \frac{1}{8}\left[-\sqrt{3}(\sqrt{5}-1)+\sqrt{10+2\sqrt{5}}\right],$$
$$\cos\frac{\pi}{15} = \frac{1}{8}\left(\sqrt{5}-1+\sqrt{3}\sqrt{10+2\sqrt{5}}\right),$$
$$\operatorname{tg}\frac{\pi}{15} = \frac{1}{4}(3-\sqrt{5})\left(2\sqrt{3}-\sqrt{10-2\sqrt{5}}\right),$$
$$\operatorname{ctg}\frac{\pi}{15} = \frac{1}{4}(1+\sqrt{5})\left(2\sqrt{3}+\sqrt{10-2\sqrt{5}}\right),$$
$$\sin\frac{2\pi}{15} = \frac{1}{8}\left[\sqrt{3}\left(\sqrt{5}+1\right)-\sqrt{10-2\sqrt{5}}\right],$$
$$\cos\frac{2\pi}{15} = \frac{1}{8}\left(\sqrt{5}+1+\sqrt{3}\sqrt{10-2\sqrt{5}}\right),$$
$$\operatorname{tg}\frac{2\pi}{15} = \frac{1}{4}(3+\sqrt{5})\left(-2\sqrt{3}+\sqrt{10+2\sqrt{5}}\right),$$
$$\operatorname{ctg}\frac{2\pi}{15} = \frac{1}{4}(\sqrt{5}-1)\left(2\sqrt{3}+\sqrt{10+2\sqrt{5}}\right).$$

11) $\sin\dfrac{\pi}{16}=\dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{2}}}$, $\quad\cos\dfrac{\pi}{16}=\dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{2}}}$,

$\operatorname{tg}\dfrac{\pi}{16}=\sqrt{2}\sqrt{2+\sqrt{2}}-(\sqrt{2}+1)$, $\quad\operatorname{ctg}\dfrac{\pi}{16}=\sqrt{2}\sqrt{2+\sqrt{2}}+(\sqrt{2}+1)$,

$\sin\dfrac{3\pi}{16}=\dfrac{1}{2}\sqrt{2-\sqrt{2-\sqrt{2}}}$, $\quad\cos\dfrac{3\pi}{16}=\dfrac{1}{2}\sqrt{2+\sqrt{2-\sqrt{2}}}$,

$\operatorname{tg}\dfrac{3\pi}{16}=\sqrt{2}\sqrt{2-\sqrt{2}}-(\sqrt{2}-1)$, $\quad\operatorname{ctg}\dfrac{3\pi}{16}=\sqrt{2}\sqrt{2-\sqrt{2}}+(\sqrt{2}-1)$.

12) $\sin\dfrac{\pi}{20}=\dfrac{\sqrt{2}}{8}\left(\sqrt{5}+1-\sqrt{10-2\sqrt{5}}\right)$,

$\cos\dfrac{\pi}{20}=\dfrac{\sqrt{2}}{8}\left(\sqrt{5}+1+\sqrt{10-2\sqrt{5}}\right)$,

$\operatorname{tg}\dfrac{\pi}{20}=\sqrt{5}+1-\sqrt{5+2\sqrt{5}}$, $\quad\operatorname{ctg}\dfrac{\pi}{20}=\sqrt{5}+1+\sqrt{5+2\sqrt{5}}$,

$\sin\dfrac{3\pi}{20}=\dfrac{\sqrt{2}}{8}\left(\sqrt{10+2\sqrt{5}}+1-\sqrt{5}\right)$,

$\cos\dfrac{3\pi}{20}=\dfrac{\sqrt{2}}{8}\left(\sqrt{10+2\sqrt{5}}-1+\sqrt{5}\right)$,

$\operatorname{tg}\dfrac{3\pi}{20}=\sqrt{5}-1-\sqrt{5-2\sqrt{5}}$, $\quad\operatorname{ctg}\dfrac{3\pi}{20}=\sqrt{5}-1+\sqrt{5-2\sqrt{5}}$.

13) $\sin\dfrac{\pi}{24}=\dfrac{1}{2}\sqrt{2-\sqrt{2+\sqrt{3}}}$, $\quad\cos\dfrac{\pi}{24}=\dfrac{1}{2}\sqrt{2+\sqrt{2+\sqrt{3}}}$,

$\operatorname{tg}\dfrac{\pi}{24}=(\sqrt{2}-1)(\sqrt{3}-\sqrt{2})$, $\quad\operatorname{ctg}\dfrac{\pi}{24}=(\sqrt{2}+1)(\sqrt{3}+\sqrt{2})$,

$\sin\dfrac{5\pi}{24}=\dfrac{1}{2}\sqrt{2-\sqrt{2-\sqrt{3}}}$, $\quad\cos\dfrac{5\pi}{24}=\dfrac{1}{2}\sqrt{2+\sqrt{2-\sqrt{3}}}$,

$\operatorname{tg}\dfrac{5\pi}{24}=(\sqrt{2}+1)(\sqrt{3}-\sqrt{2})$, $\quad\operatorname{ctg}\dfrac{5\pi}{24}=(\sqrt{2}-1)(\sqrt{3}+\sqrt{2})$.

14) $\sin\dfrac{\pi}{30}=\dfrac{1}{8}\left[-(\sqrt{5}+1)+\sqrt{3}\sqrt{10-2\sqrt{5}}\right]$,

$\cos\dfrac{\pi}{30}=\dfrac{1}{8}\left[\sqrt{3}(\sqrt{5}+1)+\sqrt{10-2\sqrt{5}}\right]$,

$\operatorname{tg}\dfrac{\pi}{30}=\dfrac{1}{4}(\sqrt{5}-1)\left(\sqrt{10+2\sqrt{5}}-2\sqrt{3}\right)$,

$\operatorname{ctg}\dfrac{\pi}{30}=\dfrac{1}{4}(3+\sqrt{5})\left(\sqrt{10+2\sqrt{5}}+2\sqrt{3}\right)$,

$\sin\dfrac{7\pi}{30}=\dfrac{1}{8}\left[-(\sqrt{5}-1)+\sqrt{3}\sqrt{10+2\sqrt{5}}\right]$,

$\cos\dfrac{7\pi}{30}=\dfrac{1}{8}\left[\sqrt{3}(\sqrt{5}-1)+\sqrt{10+2\sqrt{5}}\right]$,

$\operatorname{tg}\dfrac{7\pi}{30}=\dfrac{1}{4}(\sqrt{5}+1)\left(2\sqrt{3}-\sqrt{10-2\sqrt{5}}\right)$,

$$\operatorname{ctg}\frac{7\pi}{30} = \frac{1}{4}\,(3-\sqrt{5}\,)\left(2\sqrt{3}+\sqrt{10-2\sqrt{5}}\;\right).$$

15) $$\sin\frac{3\pi}{40} = \frac{1}{8}\left[\sqrt{2+\sqrt{2}}\,\sqrt{10-2\sqrt{5}} - \sqrt{2-\sqrt{2}}\,\left(1+\sqrt{5}\,\right)\right],$$

$$\cos\frac{3\pi}{40} = \frac{1}{8}\left[\sqrt{2-\sqrt{2}}\,\sqrt{10-2\sqrt{5}} + \sqrt{2+\sqrt{2}}\,\left(1+\sqrt{5}\,\right)\right],$$

$$\operatorname{tg}\frac{3\pi}{40} = \frac{1}{4}\left(1-\sqrt{5}+2\sqrt{2}\right)\left[\sqrt{10-2\sqrt{5}} - \sqrt{2}\,(\sqrt{5}-1)\right],$$

$$\operatorname{ctg}\frac{3\pi}{40} = \frac{1}{4}\left(-1+\sqrt{5}+2\sqrt{2}\right)\left[\sqrt{10-2\sqrt{5}} + \sqrt{2}\,(\sqrt{5}-1)\right].$$

16) $$\sin\frac{\pi}{60} = \frac{\sqrt{2}}{8}\left[\sqrt{(2+\sqrt{3}\,)(3-\sqrt{5}\,)} - \sqrt{(2-\sqrt{3}\,)(5+\sqrt{5}\,)}\;\right],$$

$$\cos\frac{\pi}{60} = \frac{\sqrt{2}}{8}\left[\sqrt{(2-\sqrt{3}\,)(3-\sqrt{5}\,)} + \sqrt{(2+\sqrt{3}\,)(5+\sqrt{5}\,)}\;\right],$$

$$\operatorname{tg}\frac{\pi}{60} = \frac{1}{4}\,(2-\sqrt{3}\,)(1+2\sqrt{3}-\sqrt{5}\,)\left(\sqrt{10-2\sqrt{5}}-2\right),$$

$$\operatorname{ctg}\frac{\pi}{60} = \frac{1}{4}\,(2+\sqrt{3}\,)(-1+2\sqrt{3}+\sqrt{5}\,)\left(\sqrt{10-2\sqrt{5}}+2\right).$$

**12.** Степенные ряды.

$$\sin x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}\,x^{2k+1}, \qquad \cos x = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\,x^{2k},$$

$$\frac{1}{\sin x} = \frac{1}{x} + 2\sum_{k=1}^{\infty}\frac{2^{2k-1}-1}{(2k)!}\,|B_{2k}|x^{2k-1} \qquad [|x|<\pi],$$

$$\frac{1}{\cos x} = \sum_{k=1}^{\infty}\frac{|E_{2k}|}{(2k)!}\,x^{2k} \qquad [|x|<\pi/2],$$

$$\operatorname{tg} x = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)}{(2k)!}\,|B_{2k}|x^{2k-1} \qquad [|x|<\pi/2],$$

$$\operatorname{ctg} x = \frac{1}{x} - \sum_{k=1}^{\infty}\frac{2^{2k}}{(2k)!}\,|B_{2k}|x^{2k-1} \qquad [|x|<\pi].$$

## I.4. Обратные тригонометрические функции

**1.** Некоторые соотношения.

Главные значения обратных тригонометрических функций определяются неравенствами

$$-\pi/2 \leqslant \arcsin x \leqslant \pi/2; \quad 0 \leqslant \arccos x \leqslant \pi \qquad [-1 \leqslant x \leqslant 1];$$

$$-\pi/2 < \operatorname{arctg} x < \pi/2; \quad 0 < \operatorname{arcctg} x < \pi \qquad [-\infty < x < \infty].$$

$\arcsin(-x) = -\arcsin x, \quad \arccos(-x) = \pi - \arccos x,$

$\operatorname{arctg}(-x) = -\operatorname{arctg} x, \quad \operatorname{arcctg}(-x) = \pi - \operatorname{arcctg} x,$

$\sin(\arcsin x) = x, \quad \cos(\arccos x) = x,$

$\operatorname{tg}(\operatorname{arctg} x) = x, \quad \operatorname{ctg}(\operatorname{arcctg} x) = x.$

$$\arcsin(\sin x) = x - 2n\pi \qquad [2n\pi - \pi/2 \leqslant x \leqslant 2n\pi + \pi/2],$$
$$= -x + (2n+1)\pi \qquad [(2n+1)\pi - \pi/2 \leqslant x \leqslant 2(n+1)\pi + \pi/2].$$
$$\arccos(\cos x) = x - 2n\pi \qquad [2n\pi \leqslant x \leqslant (2n+1)\pi],$$
$$= -x + 2(n+1)\pi \qquad [(2n+1)\pi \leqslant x \leqslant 2(n+1)\pi].$$
$$\operatorname{arctg}(\operatorname{tg} x) = x - n\pi \qquad [n\pi - \pi/2 < x < n\pi + \pi/2],$$
$$\operatorname{arcctg}(\operatorname{ctg} x) = x - n\pi \qquad [n\pi < x < (n+1)\pi].$$
$$\lim_{x\to\pm\infty} \operatorname{arctg} x = \pm\pi/2, \qquad \lim_{x\to\pm\infty} \operatorname{arcctg} x = \pi/2 \mp \pi/2.$$

**2.** Связь с логарифмической и обратными гиперболическими функциями.

$$\arcsin z = -i \ln\left(iz + \sqrt{1-z^2}\right) = -i \operatorname{Arsh}(iz),$$
$$\arccos z = -i \ln\left(z + \sqrt{z^2-1}\right) = -i \operatorname{Arch} z,$$
$$\operatorname{arctg} z = -\frac{i}{2} \ln \frac{1+iz}{1-iz} = -i \operatorname{Arth}(iz),$$
$$\operatorname{arcctg} z = -\frac{i}{2} \ln \frac{iz-1}{iz+1} = i \operatorname{Arcth}(iz).$$

**3.** Формулы дифференцирования.

$$\frac{d}{dx} \arcsin x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}},$$
$$\frac{d}{dx} \operatorname{arctg} x = \frac{1}{1+x^2}, \qquad \frac{d}{dx} \operatorname{arcctg} x = -\frac{1}{1+x^2}.$$

**4.** Связь между обратными тригонометрическими функциями.

$$\arcsin x = \frac{\pi}{2} - \arccos x,$$
$$= \arccos \sqrt{1-x^2} \qquad [0 \leqslant x \leqslant 1],$$
$$= -\arccos \sqrt{1-x^2} \qquad [-1 \leqslant x \leqslant 0],$$
$$= \operatorname{arctg} \frac{x}{\sqrt{1-x^2}} \qquad [x^2 < 1],$$
$$= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x} \qquad [0 < x \leqslant 1],$$
$$= \operatorname{arcctg} \frac{\sqrt{1-x^2}}{x} - \pi \qquad [-1 \leqslant x < 0],$$
$$= \frac{1}{2} \arcsin\left(2x\sqrt{1-x^2}\right) \qquad [x^2 \leqslant 1/2],$$
$$= \frac{\pi}{2} - \frac{1}{2} \arcsin\left(2x\sqrt{1-x^2}\right) \qquad [1/\sqrt{2} < x \leqslant 1],$$
$$= -\frac{\pi}{2} - \frac{1}{2} \arcsin\left(2x\sqrt{1-x^2}\right) \qquad [-1 \leqslant x < -1/\sqrt{2}].$$
$$\arccos x = \frac{\pi}{2} - \arcsin x,$$
$$= \arcsin \sqrt{1-x^2} \qquad [0 \leqslant x \leqslant 1],$$
$$= \pi - \arcsin \sqrt{1-x^2} \qquad [-1 \leqslant x \leqslant 0],$$

$$= \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \qquad [0 < x \leqslant 1],$$

$$= \pi + \operatorname{arctg} \frac{\sqrt{1-x^2}}{x} \qquad [-1 \leqslant x < 0],$$

$$= \operatorname{arcctg} \frac{x}{\sqrt{1-x^2}} \qquad [-1 \leqslant x \leqslant 1],$$

$$= \frac{1}{2} \arccos\left(2x^2 - 1\right) \qquad [0 \leqslant x \leqslant 1],$$

$$= \pi - \frac{1}{2} \arccos\left(2x^2 - 1\right) \qquad [-1 \leqslant x < 0].$$

$$\operatorname{arctg} x = \frac{\pi}{2} - \operatorname{arcctg} x,$$

$$= \arcsin \frac{x}{\sqrt{1+x^2}},$$

$$= \arccos \frac{1}{\sqrt{1+x^2}} \qquad [x \geqslant 0],$$

$$= -\arccos \frac{1}{\sqrt{1+x^2}} \qquad [x \leqslant 0],$$

$$= \frac{\pi}{2} - \operatorname{arctg} \frac{1}{x} \qquad [x > 0],$$

$$= -\frac{\pi}{2} - \operatorname{arctg} \frac{1}{x} \qquad [x < 0],$$

$$= \operatorname{arcctg} \frac{1}{x} \qquad [x > 0],$$

$$= \operatorname{arcctg} \frac{1}{x} - \pi \qquad [x < 0],$$

$$= \frac{\pi}{4} - \operatorname{arctg} \frac{1-x}{1+x} \qquad [x > -1],$$

$$= -\frac{3\pi}{4} - \operatorname{arctg} \frac{1-x}{1+x} \qquad [x < -1],$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} \qquad [|x| < 1],$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} + \frac{\pi}{2} \qquad [x > 1],$$

$$= \frac{1}{2} \operatorname{arctg} \frac{2x}{1-x^2} - \frac{\pi}{2} \qquad [x < -1],$$

$$= -\frac{\pi}{2} - \frac{1}{2} \arcsin \frac{2x}{1+x^2} \qquad [x \leqslant -1],$$

$$= \frac{1}{2} \arcsin \frac{2x}{1+x^2} \qquad [-1 \leqslant x \leqslant 1],$$

$$= \frac{\pi}{2} - \frac{1}{2} \arcsin \frac{2x}{1+x^2} \qquad [x \geqslant 1],$$

$$= \frac{1}{2} \arccos \frac{1-x^2}{1+x^2} \qquad [x \geqslant 0],$$

$$= -\frac{1}{2} \arccos \frac{1-x^2}{1+x^2} \qquad [x \leqslant 0].$$

$$\operatorname{arcctg} x = \frac{\pi}{2} - \operatorname{arctg} x,$$

$$= \arcsin \frac{1}{\sqrt{1+x^2}} \qquad [x>0],$$

$$= \pi - \arcsin \frac{1}{\sqrt{1+x^2}} \qquad [x<0],$$

$$= \arccos \frac{x}{\sqrt{1+x^2}},$$

$$= \operatorname{arctg} \frac{1}{x} \qquad [x>0],$$

$$= \pi + \operatorname{arctg} \frac{1}{x} \qquad [x<0].$$

**5.** Степенные ряды.

$$\arcsin x = 2\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2(2k+1)} \left(\frac{x}{2}\right)^{2k+1} \qquad [|x| \leqslant 1].$$

$$\arccos x = \frac{\pi}{2} - 2\sum_{k=0}^{\infty} \frac{(2k)!}{(k!)^2(2k+1)} \left(\frac{x}{2}\right)^{2k+1} \qquad [|x| \leqslant 1].$$

$$\operatorname{arctg} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad [|x| \leqslant 1],$$

$$= \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{-2k-1} \qquad [x>1],$$

$$= -\frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{-2k-1} \qquad [x<-1].$$

$$\operatorname{arcctg} x = \frac{\pi}{2} - \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1} \qquad [|x| < 1],$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{-2k-1} \qquad [x>1],$$

$$= \pi + \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{-2k-1} \qquad [x<-1].$$

$\mathrm{S}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^x \frac{\sin t}{\sqrt{t}}\, dt$ — синус-интеграл Френеля.

$\mathrm{C}(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^x \frac{\cos t}{\sqrt{t}}\, dt$ — косинус-интеграл Френеля.

$\mathrm{si}\,(x) = \int\limits_x^\infty \frac{\sin t}{t}\, dt$ — интегральный синус.

$\mathrm{ci}\,(x) = -\int\limits_x^\infty \frac{\cos t}{t}\, dt$ — интегральный косинус.

$\mathrm{Li}_2\,(x) = -\int\limits_0^x \frac{\ln(1-x)}{x}\, dx = \sum\limits_{k=1}^{\infty} \frac{x^k}{k^2}$ — дилогарифм Эйлера.

$C_n^m = \frac{n!}{m!(n-m)!}$ — биномиальные коэффициенты.

$C_n^0 = 1.$

$n! = 1 \cdot 2 \cdot 3 \ldots (n-1)n, \quad 0! = 1! = 1.$

$(2n)!! = 2 \cdot 4 \cdot 6 \ldots (2n) = 2^n n!,$

$(2n+1)!! = 1 \cdot 3 \cdot 5 \ldots (2n+1),$

$n!! = \begin{cases} (2k)!!, & n = 2k, \\ (2k+1)!!, & n = 2k+1, \end{cases}$

$0!! = (-1)!! = 1.$

$B_n$ — числа Бернулли.

$B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_4 = -\frac{1}{30}, \quad B_6 = \frac{1}{42},$

$B_8 = -\frac{1}{30}, \quad B_{10} = \frac{5}{66}, \quad B_{12} = -\frac{691}{2730},$

$B_{2n+1} = 0$ при $n = 1,\ 2,\ 3,\ \ldots, \quad B_n = \sum\limits_{k=0}^{n} C_n^k B_k.$

$E_n$ — числа Эйлера.

$E_0 = 1, \quad E_2 = -1, \quad E_4 = 5, \quad E_6 = -61, \quad E_8 = 1385,$

$E_{10} = -50\,521, \quad E_{12} = 2\,702\,765, \quad E_{2n+1} = 0.$

$|x| = \begin{cases} x, & x \geqslant 0, \\ -x, & x < 0. \end{cases} \quad \mathrm{sgn}\, x = \begin{cases} +1, & x > 0, \\ -1, & x < 0. \end{cases}$

$[x] = n \quad (n \leqslant x < n+1,\ n = 0,\ \pm 1,\ \pm 2,\ \ldots)$ — целая часть числа $x$.